aligning moment goes up disproportionately with load because the increase in contact area that occurs with increase in load must always occur at the perimeter of the contact patch, and it is the shear in the perimeter region that contributes the most to the moment.

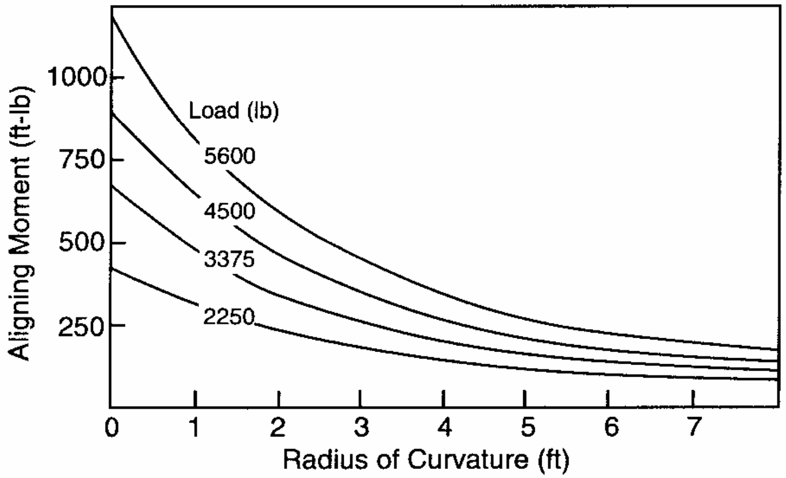

Fig. 10.21 Aligning moment as a function of radius of turn.

Relevance to Vehicle Performance

Aligning moment as a torque acting directly on the vehicle contributes a small component to the understeer of a vehicle. The fact that positive aligning moments attempt to steer the vehicle out of the turn means that they are understeer in direction. Overall, the direct action of the moments contributes only a few percent to the understeer gradient of a vehicle.

The aligning moment has a more direct influence on understeer by its action on the steered wheels. The moment is normally in the direction to turn the steered wheels out of the turn. Even though the steer deflection angles in response to aligning moments may be small (fractions of a degree in normal driving), this is normally an important contribution to understeer gradient.

Aligning moment is also important to steering feel for a moving vehicle. Its contribution is equal to or greater than caster angle in producing returnability torques—the torques acting to return the steering to the straight-ahead position when cornering.

Aligning moment arising from path curvature is primarily important for static steer and very low-speed maneuvering. The moment is the dominant source of steering torque and may be quite large. Because this represents a condition that places highest torque demand on a steering system it must be considered in sizing power-steering hardware and in durability testing. By offsetting the tire outside of the steering axis (positive scrub) the tire can be allowed to roll on a radius that will decrease the magnitude of the aligning moment in static steer situations. In operating situations, simply moving the vehicle at low speed while increasing the steering angle greatly increases the radius of curvature and thereby reduces the steering torque required.

COMBINED BRAKING AND CORNERING

When a tire is operated under conditions of simultaneous longitudinal and lateral slip, the respective forces depart markedly from those values derived under independent conditions. The application of longitudinal slip generally tends to reduce the lateral force at a given slip angle condition, and conversely, application of slip angle reduces the longitudinal force developed under a given braking condition. This behavior is shown in Figure 10.22.

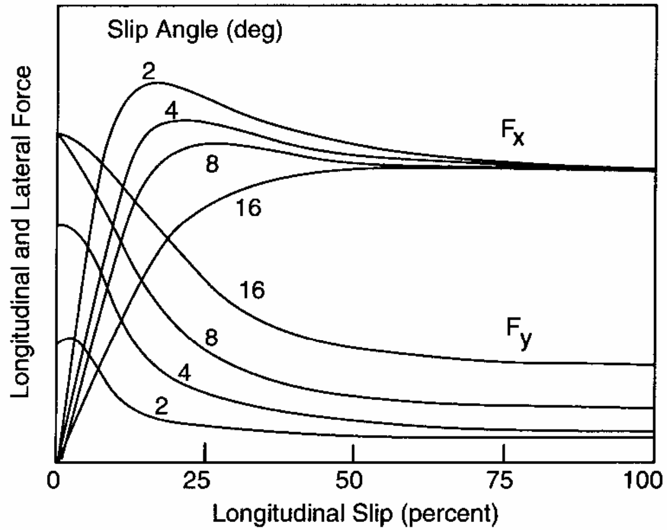

Fig. 10.22 Brake and lateral forces as a function of longitudinal slip.

Friction Circle

The general effect on lateral force when braking is applied is illustrated in the traction field of Figure 10.23. The individual curves represent the lateral force at a given slip angle. As the brake force is applied, the lateral force gradually diminishes due to the additional slip induced in the contact area from the braking demand.

This type of display of a tire traction field is the basis for the "friction circle" (or friction ellipse) concept [11]. Recognizing that the friction limit for a tire, regardless of direction, will be determined by the coefficient of friction times the load, it is clear that the friction can be used for lateral force, or brake force, or a combination of the two, in either the positive or negative directions. But, in no case can the vector total of the two exceed the friction limit. The limit is therefore a circle in the plane of the lateral and longitudinal forces. The portion of the circle in the figure is the friction circle for the positive quadrant of the traction field. The limit is characterized as a friction circle for tires which

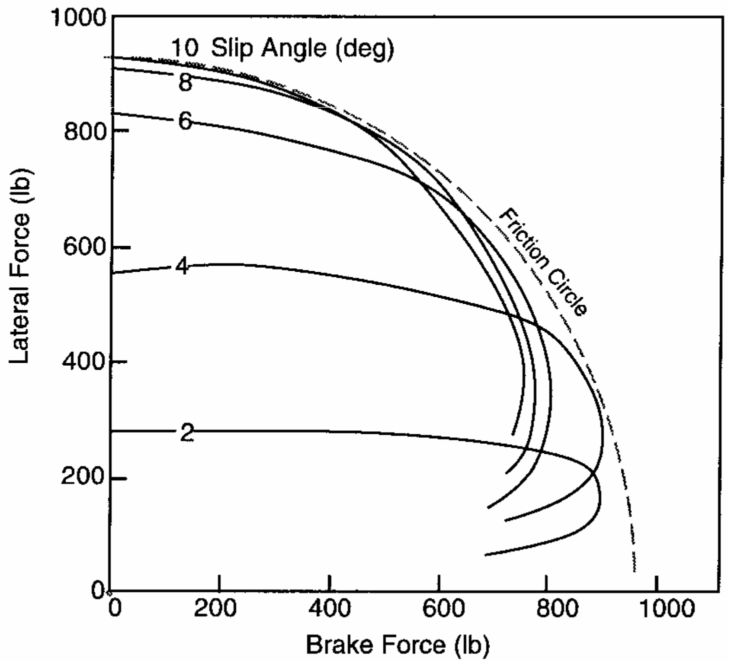

Fig. 10.23 Lateral force versus longitudinal force at constant slip angles.

have effectively the same limits for lateral and braking forces. Certain specialized tires, however, may be optimized for lateral traction or braking traction, in which case the limit is not a circle but an ellipse.

The friction circle concept has been used in recent years as a means to evaluate race car drivers by making continuous recordings of the lateral and longitudinal accelerations maintained on a track. For maximum efficiency in getting around a closed course, the tires should be working continuously at either the cornering or braking/acceleration limits. Therefore, the combined lateral and longitudinal accelerations measured on the car should always be pressing the friction limit, and the most effective driver is the one who can most closely maintain this optimum. By plotting the record of the two accelerations on a polar plot similar to Figure 10.23, one gets a visual indication of the performance of the driver by observing the percentage of time spent at the friction limit.

Figure 10.23 illustrates another observation that is frequently made under conditions of combined traction. Note that at intermediate slip angle conditions near 4 degrees, the application of moderate levels of brake force actually increases the lateral force developed at that slip angle. This phenomenon is shown more precisely in the plot of Figure 10.24, which shows the lateral force and aligning moment under tractive forces in both the braking and driving directions [12].

Using the free-rolling (zero tractive force) values of lateral force as a reference point, it is seen that when braking (negative) force is applied, the lateral force increases slightly while the aligning moment decreases. In effect, the presence of the braking force acts to stiffen the tire structure (sidewalls and/or tread) with respect to the mechanism that generates lateral force. The reduction of aligning moment implies a significant redistribution of the shear forces in the contact patch. As the braking force increases toward its maximum value the lateral force diminishes because the friction limits are being approached. Concurrently, the aligning moment decreases to the point where it may actually go negative near the braking limit. A negative aligning moment attempts to steer the wheel to a greater slip angle, and may adversely affect stability in braking, particularly through its effects on the steering system [13].

Under a moderate driving (positive) traction force the opposite effects are observed. Lateral force decreases slightly, although aligning moment increases markedly. At levels near the friction limit both lateral force and aligning moment decrease. Unlike braking, however, the aligning moment never goes negative near the limit of driving force.

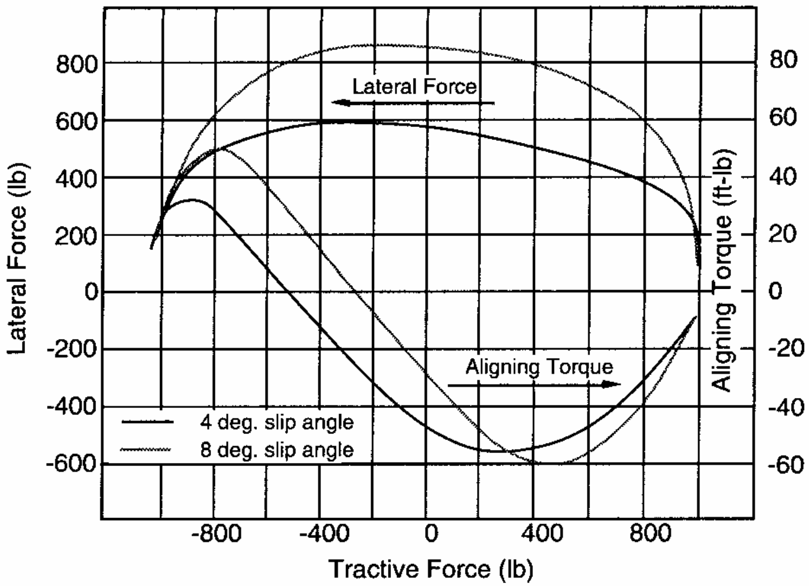

Fig. 10.24 Lateral force and aligning torque versus tractive force.

Variables

Although tire type (radial versus bias-ply) and inflation pressure have significant influences on cornering stiffness, behavior under combined slip is qualitatively similar to that shown above. The behavior is insensitive to velocity, and is only affected by surface conditions through the influence on friction limits.

Relevance to Vehicle Performance

The combined slip behavior of tires is only meaningful in the context of braking-in-a-turn maneuvers. When brakes are applied to a vehicle in a steady turn, the increasing level of tire longitudinal slip produces a loss in tire side force which characteristically serves to disturb the path and/or yaw orientation of the vehicle. Alternatively, if a large steering input is applied while the vehicle is braking, both the braking performance and the cornering performance stand to be degraded in comparison to the performances expected with independent inputs of steering or braking.

Minimal degradation of braking performance occurs with concurrent cornering up to levels of about 0.3 g lateral acceleration. However, as limit braking is approached, the directional or yaw response can be degraded to the point of total loss of control. The nature of the control loss will depend on the order in which front and rear tires approach the wheel-lock condition. Front-wheel lockup will render the vehicle unsteerable, whereas rear-wheel lockup precipitates spinout.

CONICITY AND PLY STEER

The behavior of tires in the near-zero lateral slip region has grown more important in recent years with the refinement of high-speed automobiles. The importance derives from the emphasis that is put on on-center feel of the steering system.

For an ideal tire, zero lateral force coincides with zero slip angle, but for actual tires this is not true. For actual tires the behavior of lateral force at small slip angles will be similar to that shown in Figure 10.25. In this plot the lateral force is plotted as the tire is rolled in both directions, arbitrarily labeled as forward and reverse in the figure. The important observation in the figure is that the lateral force behavior differs with the direction of rotation (forward versus reverse) and may be offset from the origin of the graph.

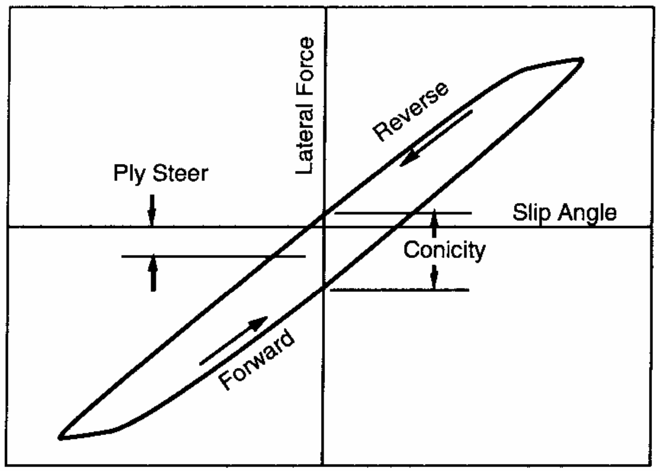

Fig. 10.25 Lateral force behavior around zero slip angle.

One mechanism in the tire accounting for this behavior is conicity in the construction. Conicity derives from small side-to-side differences in the tire such as an asymmetrical offset in the positioning of the belt. As the name implies, these variations are manifest in a tire as a bias toward a conical shape as illustrated in Figure 10.26. Because of this shape, a freely rolling tire will want to follow an arc centered about the apex of the cone, shown at the right of Figure 10.26. Forced to follow a straight line this tire will experience a lateral force toward the right in the figure, regardless of which direction it may roll. By the SAE convention, when the tire rolls upward in the top view the lateral force is to the right and is positive in direction. Rolling downward, the force will again be to the right, but since the longitudinal axis of the tire is now pointed downward it is a negative lateral force. Thus conicity is manifest as a difference in the lateral force at zero slip angle when the tire is rolled in opposite directions. Conicity has the character of being random in direction and is dependent on quality control in tire construction. Turning a tire on the rim will change the direction of the lateral force caused by conicity.

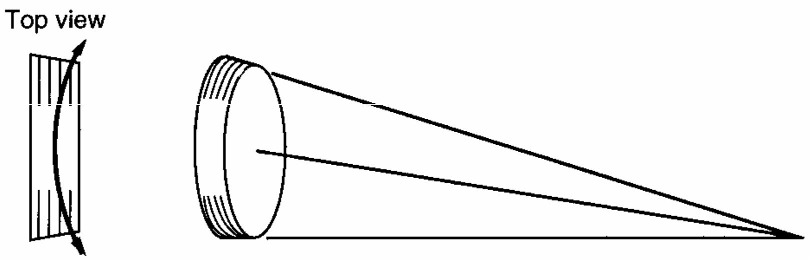

Fig. 10.26 Conicity in a pneumatic tire.

The other mechanism that may be present in a tire is ply steer, which arises from the angle of the cords in the belt layers. To avoid this bias in the tire's preferred rolling direction, belts are constructed with alternating belt layers at opposite angles, but a perfect balance is impossible to achieve. Thus a free-rolling tire will exhibit a tendency to drift from its direction of heading. Instead of following an arc as illustrated in Figure 10.26, it will follow a line that is skewed with respect to its center plane. If, when rolling in one direction, it creates a lateral force to the right in the SAE tire axis system, when rolled in the opposite direction, it will again exhibit a force to the right in the SAE tire axis system. Thus when tire lateral force properties are measured in the vicinity of zero-degree slip angle, ply steer is manifest as a non-zero offset in the lateral

force averaged from both directions of travel. Ply steer is dependent on tire design; hence, it will be nearly equal in magnitude and direction for all tires of a common design. Turning the tire on the rim does not change the direction of the lateral force caused by ply steer.

Both conicity and ply steer force magnitudes are dependent on the vertical load carried by a tire. Conicity is more sensitive to inflation pressure and may be reduced by making adjustments to the pressure.

Relevance to Vehicle Performance

The effects of conicity and ply steer are to create a "pull" in the steering system or a "drift" in the tracking of the car. Pull refers to a condition where the driver must apply a continuous torque to the steering wheel holding it off-center to maintain the vehicle on a straight path; or with the wheel free and in the center position, the vehicle will follow a curved path.

Excess conicity on the front wheels may cause the steering to pull to such a degree that it is fatiguing to the driver and becomes a source of customer dissatisfaction. Conicity on the rear wheels will cause the vehicle to track with the rear wheels offset from the front. It may also affect the centering of the steering wheel in the straight-ahead position.

Since ply steer is likely to act on all wheels in the same direction, a vehicle may exhibit a slight drift due to ply steer forces. This may require some steering wheel offset to compensate and keep the vehicle traveling straight; no steering pull is likely.

DURABILITY FORCES

Although tires act as a cushion between the vehicle and the road, the bumps present in most roads transmit forces that are perceptible to the motorist and may contribute to the cyclic loading and fatigue of suspension components. Road bump features that have a characteristic length on the same order as the length of the tire contact patch (e.g., tar strips, faults in concrete surfaces, potholes) generate forces that are dependent on the tire's ability to envelop these features. Because these forces, particularly those attributable to encounters with potholes, may be large in magnitude and thus significant to fatigue and durability of a vehicle, they are often referred to as durability forces.

A smooth road surface is seen as a flat plane by a free-rolling tire and generates predominantly vertical force inputs to the tire. Even at high speed the vertical inputs are slow to change in magnitude relative to the time it takes the tire to advance along the length of its contact patch. However, when a tire encounters an abrupt discontinuity in a road surface, such as the edge of a pothole, the forces may change markedly as the feature passes through the contact patch. Dynamic vertical and longitudinal forces are generated in the process. The shape and magnitude of the force inputs depend on the properties and mechanics of the tire.

Tire performance when enveloping road discontinuities has been studied by examining performance when tires negotiate small step changes in road elevation [14]. (In theory, any shape of bump can be approximated by a judiciously selected combination of small steps.) Mechanistically the tire can be thought of as a series of radial springs (sometimes modeled with dampers in parallel with each spring) in contact with the surface, to understand its behavior. When a tire encounters an upward step in the road surface, the vertical and longitudinal forces on the tire will change abruptly as shown in Figure 10.27.

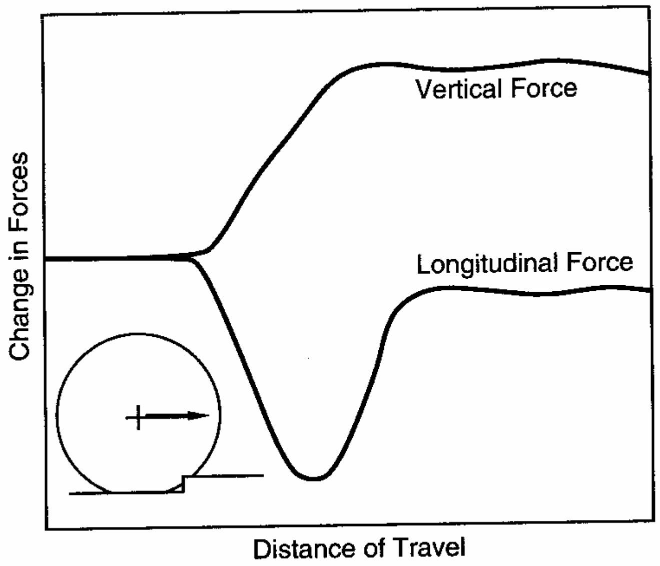

Fig. 10.27 Vertical and longitudinal tire forces produced by a step in the road.

As the tire contacts the leading edge of the bump, the vertical force begins to rise due to depression of the tread. The force rises more or less continuously until the full contact patch has advanced onto the bump. Performed at high speed, the axle on which the tire is mounted is not able to respond and move upward in the time it takes the entire contact patch to move onto the bump. Thus an increase in vertical load occurs that is approximately equal to the bump height times the vertical stiffness of the tire.

The encounter with the bump also creates a longitudinal force as a result of several mechanisms. For the tire to rise onto the bump, a longitudinal force is required. This force must be provided by the axle, thus a negative (opposite to the direction of travel) force is experienced when the tire first encounters the edge of the bump. Once on the bump the increased vertical load causes an increase in the rolling resistance of the tire, so the longitudinal force does not immediately return to its original value, but must wait for the axle to adjust to a new height representing the balance of vertical forces.

At high speed, a second mechanism is at work as well. With the change in effective radius of the tire on the bump, it must assume a new rotation rate in correspondence with its forward speed. With a smaller radius, the tire must increase its rotational speed. This is accomplished by generating a shear force in the contact patch which is ultimately balanced out by an opposite force reaction at the axle. To speed up the rotation rate of the tire, a second component of negative force is imposed on the axle. The rate at which the tire mounts the bump, the rate at which it must speed up, and the magnitude of the longitudinal force created depends on the forward speed of the wheel. Thus this component of longitudinal force depends on the speed of travel.

When a tire encounters a downward step in the road surface, similar forces are created differing only in their direction of action. That is, a downward step causes a decrease in vertical force and creates an impulsive force in the forward direction on the axle.

TIRE VIBRATIONS

Thus far the tire has been treated as a mechanism for generating forces by which a vehicle may be controlled in braking and turning. With regard to ride dynamics it is seen to behave primarily as a spring which absorbs the roughness features in the road and interacts with the vertical motions of the body and unsprung masses. The tire, however, is also a dynamic system with resonances which affect the transmission of vibrations to the vehicle and may interact with vehicle resonances [15].

A relatively large portion of the tire mass is concentrated in the tread which is connected to the wheel by the compliant sidewalls. This combination of mass and compliance permits the tread to resonate when excited by road inputs. Figure 10.28 shows examples of the first three modal resonances of the tire in the vertical plane.

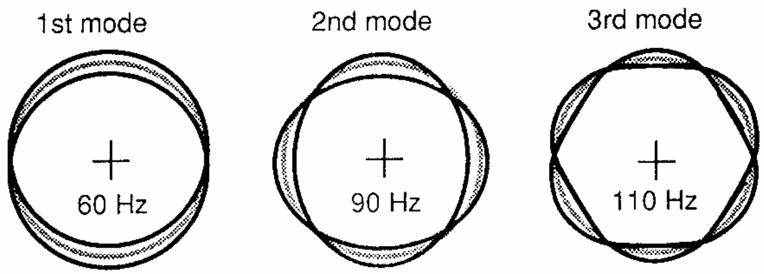

Fig. 10.28 First, second and third modal resonances of a tire.

The first mode, which will occur somewhere near 60 Hz for a passenger-car tire, involves a simple vertical motion of the entire tread band without distortion. The mode is easily excited by vertical input at the proper frequency in the contact patch. Since the entire tread band moves up and down in unison, the force associated with the resonance is transmitted to the wheel and axle.

The second mode contrasts with the first in that the tread band is oscillating in an elliptical fashion always remaining symmetrical about the vertical and horizontal axes. The top and bottom of the tread are always moving out of phase so that no net vertical force is imposed on the wheel. (Likewise, there is no net fore-aft force.) Although the resonance can be excited by vertical inputs at the contact patch, the tire is very effective at absorbing the inputs without transmitting forces to the axle. In a similar fashion the third- and higher-order resonances of the tire are very effective in absorbing road inputs without transmitting them to the wheel and axle.

In between these modal resonances the tire has anti-resonant modes characterized by very asymmetrical tread distortion and little mobility at the contact patch. The asymmetry of the motion results in unbalanced forces being imposed around the circumference of the wheel, such that the resultant force is transmitted to the wheel. The fact that the contact patch is stationary implies that the tire appears as a very stiff, rather than compliant, element with regard to road inputs at this frequency.

From this simple picture of a tire as a resonant system, it is possible to begin building an understanding of the dynamic behavior of the tire in transmitting road vibrations in the chassis of a motor vehicle. The system can be characterized by examining several relevant properties. Figure 10.29 shows experimental measurements on a radial tire mounted on a passenger car exposed to vertical excitation at the contact patch [15].

The transmissibility in this figure is defined as the ratio of acceleration on the axle per unit of road displacement at the contact patch. The first peak just below 20 Hz is axle hop resonance in which the tire acts as the primary stiffness constraining the unsprung mass. More of interest are the several peaks at higher frequencies. Note that they occur at frequencies in between the tire resonances (i.e., the anti-resonant points), corresponding to the peaks in transmissibility and peaks as well in the footprint stiffness of the tire (tire force per unit of road displacement).

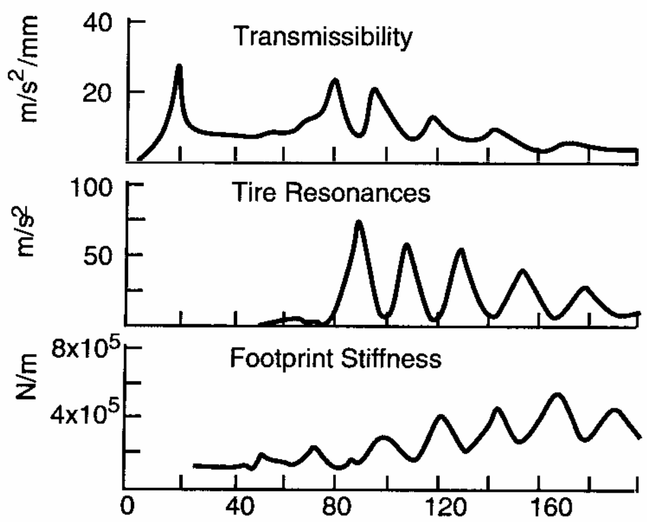

Fig. 10.29 Tire resonance properties measured on a vehicle.

This somewhat simplified picture of tire dynamics hints at the forces that will be input at the wheel of a motor vehicle. Figure 10.30 shows the spectra of forces measured when a passenger-car tire encounters a small obstacle at a speed of 30 feet per second [16]. Data are shown for both a radial and bias-ply tire and for forces in both the vertical and longitudinal directions.

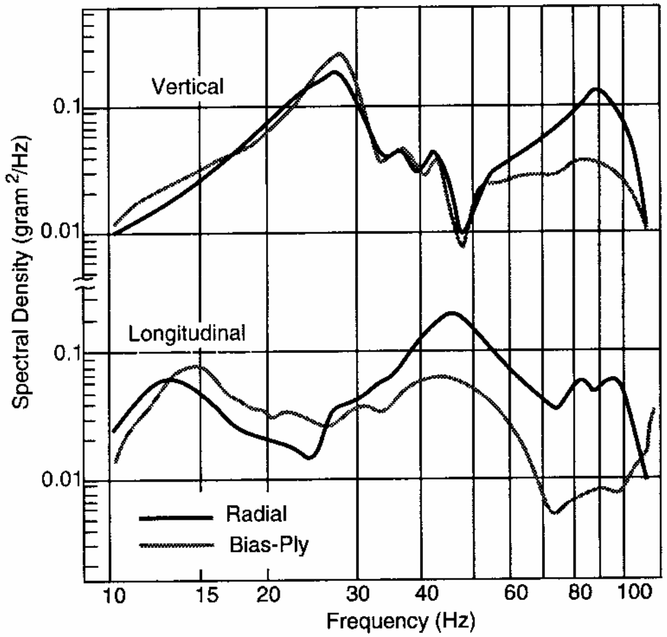

Fig. 10.30 Spectra of forces measured when a tire encounters an obstacle.

In the vertical direction the radial tire is distinguished by the increased amplitude of force in the frequency range of 50 to 100 Hz. From the previous discussion it would be expected that this behavior is a result of the high transmissibility of the anti-resonant modes in this frequency range for radial-ply tires. Obviously, bias-ply tires are much better in this range.

Perhaps the more important distinction between the two types of tires is seen in the spectra of longitudinal response. Except for a narrow band around 15 to 20 Hz, radial tires are more responsive in the longitudinal direction than bias-ply tires. At the higher frequencies the greater transmissibility indicates a higher effective stiffness in the longitudinal direction. The higher transmissibility of the radial tire near 10 Hz is one of the key differences that required "ride tuning" when radials were first introduced to the American automotive market. On vehicles historically developed on bias-ply tires, the application of radials produced more suspension fore/aft vibration requiring the addition of more longitudinal compliance in suspensions to prevent these vibrations from being transmitted to the body.

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Appendix A
SAE J670e
Vehicle Dynamics Terminology

SAE Recommended Practice
Issued by the Vehicle Dynamics Committee July 1952
Last revised July 1976

NOTE: Italicized words and phrases appearing in a definition are themselves defined elsewhere in this Terminology.

1. Mechanical Vibration-Qualitative Terminology

1.1 Vibration (Oscillation), General — Vibration is the variation with time of the displacement of a body with respect to a specified reference dimension when the displacement is alternately greater and smaller than the reference. (Adapted from ANS Z24.1-1951, item 1.040.)

1.2 Free Vibration — Free vibration of a system is the *vibration* during which no variable force is externally applied to the system. (Adapted from ANS Z24.1-1951, item 2.135.)

1.3 Forced Vibration — Forced vibration of a system is *vibration* during which variable forces outside the system determine the period of the vibration. (Adapted from ANS Z24.1-1951, item 2.130.)

1.3.1 RESONANCE — A *forced vibration* phenomenon which exists if any small change in *frequency* of the applied force causes a decrease in the *amplitude* of the vibrating system. (Adapted from ANS Z24.1-1951, item 2.105.)

1.4 Self-Excited Vibration — *Vibrations* are termed self-excited if the vibratory motion produces cyclic forces which sustain the *vibration.*

1.5 Simple Harmonic Vibration — *Vibration* at a point in a system is simple harmonic when the displacement with respect to time is described by a simple sine function.

1.6 Steady-State Vibration — Steady-state vibration exists in a system if the displacement at each point recurs for equal increments of time. (Adapted from ANS Z24.1-1951, items 11.005 and 1.045.)

1.7 Periodic Vibration — Periodic vibration exists in a system when recurring *cycles* take place in equal time intervals.

1.8 Random Vibration — Random vibration exists in a system when the *oscillation* is sustained but irregular both as to *period* and *amplitude.*

1.9 Transient Vibration — Transient vibration exists in a system when one or more component *oscillations* are discontinuous.

2. *Mechanical Vibration-Quantitative Terminology*

2.1 Period — Period of an *oscillation* is the smallest increment of time in which one complete sequence of variation in displacement occurs. (Adapted from ANS Z24.1-1951, item 1.050.)

2.2 Cycle — Cycle of *oscillation* is the complete sequence of variations in displacement which occur during a *period.* (Adapted from ANS Z24.1-1951, item 1.055.)

2.3 Frequency — Frequency of *vibration* is the number of *periods* occurring in unit time. (Adapted from ANS Z24.1-1951, item 1.060.)

2.3.1 NATURAL FREQUENCY — Natural frequency of a body or system is a frequency of free vibration. (Same as ANS Z24.1-1951, item 2.140.)

2.3.2 EXCITING FREQUENCY — Exciting frequency is the *frequency* of variation of the exciting force.

2.3.3 FREQUENCY RATIO — The ratio of *exciting frequency* to the *natural frequency.*

2.3.4 RESONANT FREQUENCY — *Frequency* at which *resonance* exists. (Same as ANS Z24.1-1951, item 2.110.)

2.4 Amplitude — Amplitude of displacement at a point in a *vibrating system* is the largest value of displacement that the point attains with reference to its equilibrium position. (Adapted from ANS Z24.1-1951, item 1.070.)

2.4.1 PEAK-TO-PEAK AMPLITUDE (DOUBLE AMPLITUDE) — Peak-to-

Peak amplitude of displacement at a point in a *vibrating system* is the sum of the extreme values of displacement in both directions from the equilibrium position. (Adapted from ANS Z24.1-1951, item 1.075.)

2.4.2 STATIC AMPLITUDE — Static amplitude in *forced vibration* at a point in a system is that displacement of the point from its specified equilibrium position which would be produced by a static force equal to the maximum value of exciting force.

2.4.3 AMPLITUDE RATIO (RELATIVE MAGNIFICATION FACTOR) — The ratio of a *forced vibration* amplitude to the *static amplitude.*

2.5 Velocity — Velocity of a point in a *vibrating system* is the time rate of change of its displacement. (Adapted from ANS Z24.1-1951, item 1.345.)

In *simple harmonic vibration,* the maximum velocity,

$$v_m = \omega x$$

where:

ω = $2\pi f$
f = frequency
x = amplitude

2.6 Acceleration — Acceleration of a point is the time rate of change of the *velocity* of the point. (Same as ANS Z24.1-1951, item 1.355.)

In *simple harmonic vibration*, the maximum acceleration,

$$a_m = \omega^2 x$$

2.7 Jerk — "Jerk" is a concise term used to denote the time rate of change of *acceleration* of a point.

In *simple harmonic motion,* the maximum jerk,

$$j_m = \omega^3 x$$

2.8 Transmissibility — Transmissibility in *forced vibration* is the ratio of the transmitted force to the applied force.

3. Vibrating Systems

3.1 Degree of Freedom — The number of degrees of freedom of a *vibrating* system is the sum total of all ways in which the masses of the system can be independently displaced from their respective equilibrium positions.

EXAMPLES — A single rigid body constrained to move only vertically on supporting springs is a system of one degree of freedom. If the same mass is also permitted angular displacement in one vertical plane, it has two degrees of freedom; one being vertical displacement of the center of gravity; the other, angular displacement about the center of gravity.

3.2 Linear — Linear *vibrating systems* are those in which all the variable forces are directly proportional to the displacement, or to the derivatives of the displacement, with respect to time.

3.3 Nonlinear — Nonlinear *vibrating systems* are those in which any of the variable forces are not directly proportional to the displacement, or to its derivatives, with respect to time.

EXAMPLE — A system having a variable *spring rate*.

3.4 Undamped — Undamped systems are those in which there are no forces opposing the vibratory motion to dissipate energy.

3.5 Damped — Damped systems are those in which energy is dissipated by forces opposing the vibratory motion.

Any means associated with a *vibrating system* to balance or modulate exciting forces will reduce the vibratory motion, but are not considered to be in the same category as damping. The latter term is applied to an inherent characteristic of the system without reference to the nature of the excitation.

3.5.1 VISCOUS DAMPING — Damping in which the force opposing the motion is proportional and opposite in direction to the velocity.

3.5.2 CRITICAL DAMPING — The minimum amount of *viscous damping* required in a *linear system* to prevent the displacement of the system from passing the equilibrium position upon returning from an initial displacement.

3.5.3 DAMPING RATIO — The ratio of the amount of *viscous damping* present in a system to that required for *critical damping*.

3.5.4 COULOMB DAMPING — Damping in which a constant force opposes

the vibratory motion.

3.5.5 COMPLEX DAMPING — Damping in which the force opposing the vibratory motion is variable, but not proportional, to the *velocity*.

In the field of aircraft flutter and vibration, complex damping is also used to denote a specific type of damping in which the damping force is assumed to be *harmonic* and in phase with the *velocity* but to have an *amplitude* proportional to the *amplitude* of displacement.

4. Components and Characteristics of Suspension Systems

4.1 Vibrating Mass and Weight

4.1.1 SPRUNG WEIGHT — All weight which is supported by the suspension, including portions of the weight of the suspension members.

In the case of most vehicles, the sprung weight is commonly defined as the total weight less the weight of *unsprung* parts.

4.1.2 SPRUNG MASS — Considered to be a rigid body having equal mass, the same center of gravity, and the same moments of inertia about identical axes as the total *sprung weight*.

4.1.3 DYNAMIC INDEX — (k^2/ab ratio) is the square of the radius of gyration (k) of the *sprung mass* about a transverse axis through the center of gravity, divided by the product of the two longitudinal distances (a and b) from the center of gravity to the front and rear *wheel centers*.

4.1.4 UNSPRUNG WEIGHT — All weight which is not carried by the suspension system, but is supported directly by the tire or wheel, and considered to move with it.

4.1.5 UNSPRUNG MASS — The unsprung masses are the equivalent masses which reproduce the inertia forces produced by the motions of the corresponding unsprung parts.

4.2 Spring Rate — The change of load of a spring per unit deflection, taken as a mean between loading and unloading at a specified load.

4.2.1 STATIC RATE — Static rate of an elastic member is the rate measured between successive stationary positions at which the member has settled to substantially equilibrium condition.

4.2.2 DYNAMIC RATE — Dynamic rate of an elastic member is the rate measured during rapid deflection where the member is not allowed to reach static equilibrium.

4.3 Resultant Spring Rate

4.3.1 SUSPENSION RATE (WHEEL RATE) — The change of wheel load, at the *center of tire contact,* per unit vertical displacement of the *sprung mass* relative to the wheel at a specified load.

If the *wheel camber* varies, the displacement should be measured relative to the lowest point on the rim centerline.

4.3.2 TIRE RATE (STATIC) — The *static rate* measured by the change of wheel load per unit vertical displacement of the wheel relative to the ground at a specified load and inflation pressure.

4.3.3 RIDE RATE — The change of wheel load, at the *center of tire contact,* per unit vertical displacement of the *sprung mass* relative to the ground at a specified load.

4.4 Static Deflection

4.4.1 TOTAL STATIC DEFLECTION — Total static deflection of a loaded suspension system is the overall deflection under the static load from the position at which all elastic elements are free of load.

4.4.2 EFFECTIVE STATIC DEFLECTION — Effective static deflection of a loaded suspension system equals the static load divided by the *spring rate* of the system at that load.

Total static deflection and effective static deflection are equal when the *spring rate* is constant.

4.4.3 SPRING CENTER — The vertical line along which a vertical load applied to the sprung mass will produce only uniform vertical displacement.

4.4.3.1 Parallel Springing — Describes the suspension of a vehicle in which the *effective static deflections* of the two ends are equal; that is, the *spring center* passes through the center of gravity of the *sprung mass.*

4.5 Damping Devices — As distinct from specific types of damping, damping devices refer to the actual mechanisms used to obtain damping of suspension systems.

4.5.1 SHOCK ABSORBER — A generic term which is commonly applied to hydraulic mechanisms for producing damping of suspension systems.

4.5.2 SNUBBER — A generic term which is commonly applied to mechanisms which employ dry friction to produce damping of suspension systems.

5. Vibrations of Vehicle Suspension Systems

5.1 Sprung Mass Vibrations

5.1.1 RIDE — The *low frequency* (up to 5 Hz) *vibrations* of the sprung mass as a rigid body.

5.1.1.1 Vertical (Bounce) — The translational component of ride *vibrations* of the *sprung mass* in the direction of the vehicle z-axis. (See Fig. 2.)

5.1.1.2 Pitch — The angular component of ride *vibrations* of the *sprung mass* about the vehicle y-axis.

5.1.1.3 Roll — The angular component of ride *vibrations* of the *sprung mass* about the vehicle x-axis.

5.1.2 SHAKE — The intermediate *frequency* (5-25 Hz) *vibrations* of the *sprung mass* as a flexible body.

5.1.2.1 Torsional Shake — A mode of *vibration* involving twisting deformations of *sprung mass* about the vehicle x-axis.

5.1.2.2 Beaming — A mode of *vibration* involving predominantly bending deformations of the *sprung mass* about the vehicle y-axis.

5.1.3 HARSHNESS — The high frequency (25-100 Hz) *vibrations* of the structure and/or components that are perceived tactually and/or audibly.

5.1.4 BOOM — A high intensity *vibration* (25-100 Hz) perceived audibly and characterized as sensation of pressure by the ear.

5.2 Unsprung Mass Vibrations

5.2.1 WHEEL VIBRATION MODES

5.2.1.1 Hop — The vertical oscillatory motion of a wheel between the road surface and the *sprung mass.*

5.2.1.1.1 Parallel hop is the form of wheel hop in which a pair of wheels hop in phase.

5.2.1.1.2 Tramp is the form of wheel hop in which a pair of wheels hop in opposite phase.

5.2.1.2 Brake Hop — An oscillatory hopping motion of a single wheel or of a pair of wheels which occurs when brakes are applied in forward or reverse motion of the vehicle.

5.2.1.3 Power Hop — An oscillatory hopping motion of a single wheel or of a pair of wheels which occurs when *tractive force* is applied in forward or reverse motion of the vehicle.

5.2.2 AXLE VIBRATION MODES

5.2.2.1 Axle Side Shake — Oscillatory motion of an axle which consists of transverse displacement.

5.2.2.2 Axle Fore-and-Aft Shake — Oscillatory motion of an axle which consists purely of longitudinal displacement.

5.2.2.3 Axle Yaw — Oscillatory motion of an axle around the vertical axis through its center of gravity.

5.2.2.4 Axle Windup — Oscillatory motion of an axle about the horizontal transverse axis through its center of gravity.

5.2.3 STEERING SYSTEM VIBRATIONS

5.2.3.1 Wheel Flutter — Forced *oscillation* of steerable wheels about their steering axes.

5.2.3.2 Wheel Wobble — A self-excited *oscillation* of steerable wheels about their steering axes, occurring without appreciable *tramp.*

5.2.3.3 Shimmy — A self-excited *oscillation* of a pair of steerable wheels about their steering axes, accompanied by appreciable *tramp.*

5.2.3.4 Wheelfight — A rotary disturbance of the steering wheel produced by forces acting on the steerable wheels.

6. *Suspension Geometry*

6.1 Kingpin Geometry

6.1.1 WHEEL PLANE — The central plane of the tire, normal to the *spin axis.*

6.1.2 WHEEL CENTER — The point at which the *spin axis* of the wheel intersects the *wheel plane.*

6.1.3 CENTER OF TIRE CONTACT — The intersection of the *wheel plane* and the vertical projection of the *spin axis* of the wheel onto the road plane. (See Note 1.)

6.1.4 KINGPIN INCLINATION — The angle in front elevation between the steering axis and the vertical.

6.1.5 KINGPIN OFFSET — Kingpin offset at the ground is the horizontal distance in front elevation between the point where the steering axis intersects the ground and the *center of tire contact.*

The kingpin offset at the *wheel center* is the horizontal distance in front elevation from the *wheel center* to the steering axis.

6.2 Wheel Caster

6.2.1 CASTER ANGLE — The angle in side elevation between the steering axis and the vertical. It is considered positive when the steering axis is inclined rearward (in the upward direction) and negative when the steering axis is inclined forward.

6.2.2 RATE OF CASTER CHANGE — The change in *caster angle* per unit vertical displacement of the *wheel center* relative to the *sprung mass.*

6.2.3 CASTER OFFSET — The distance in side elevation between the point where the steering axis intersects the ground, and the *center of tire contact.* The offset is considered positive when the intersection point is forward of the tire contact center and negative when it is rearward.

6.2.4 CENTRIFUGAL CASTER — The unbalance moment about the steering axis produced by a lateral acceleration equal to gravity acting at the combined center of gravity of all the steerable parts. It is considered positive if the combined center of gravity is forward of the steering axis and negative if rearward of the steering axis.

6.3 Wheel Camber

6.3.1 CAMBER ANGLE — The inclination of the *wheel plane* to the vertical. It is considered positive when the wheel leans outward at the top and negative when it leans inward.

6.3.2 RATE OF CAMBER CHANGE — The change of *camber angle* per unit vertical displacement of the *wheel center* relative to the *sprung mass.*

6.3.2.1 Swing Center — That instantaneous center in the transverse vertical plane through any pair of *wheel centers* about which the wheel moves relative to the *sprung mass.*

6.3.2.2. Swing-Arm Radius — The horizontal distance from the *swing center* to the *center of tire contact.*

6.3.3 WHEEL TRACK (WHEEL TREAD) — The lateral distance between the *center of tire contact* of a pair of wheels. For vehicles with dual wheels, it is the distance between the points centrally located between the *centers of tire contact* of the inner and outer wheels. (See SAE J693.)

6.3.4 TRACK CHANGE — The change in *wheel track* resulting from vertical suspension displacements of both wheels in the same direction.

6.3.5 RATE OF TRACK CHANGE — The change in *wheel track* per unit vertical displacement of both *wheel centers* in the same direction relative to the *sprung mass.*

6.4 Wheel Toe

6.4.1 STATIC TOE ANGLE (DEG) — The static toe angle of a wheel, at a specified wheel load or relative position of the *wheel center* with respect to the *sprung mass,* is the angle between a longitudinal axis of the vehicle and the line of intersection of the *wheel plane* and the road surface. The wheel is "toed-in" if the forward portion of the wheel is turned toward a central longitudinal axis of the vehicle, and "toed-out" if turned away.

6.4.2 STATIC TOE (IN [MM]) — Static toe-in or toe-out of a pair of wheels, at a specified wheel load or relative position of the *wheel center* with respect to the *sprung mass,* is the difference in the transverse distances between the *wheel planes*, taken at the extreme rear and front points of the tire treads. When the distance at the rear is greater, the wheels are "toed-in" by this amount; and where smaller, the wheels are "toed-out." (See Note 2.)

6.5 Compression — The relative displacement of *sprung* and *unsprung masses* in the suspension system in which the distance between the masses

decreases from that at static condition.

6.5.1 RIDE CLEARANCE — The maximum displacement in *compression* of the *sprung mass* relative to the *wheel center* permitted by the suspension system, from the normal load position.

6.5.2 METAL-TO-METAL POSITION (COMPRESSION) — The point of maximum *compression* travel limited by interference of substantially rigid members.

6.5.3 BUMP STOP — An elastic member which increases the *wheel rate* toward the end of the *compression* travel.

The bump stop may also act to limit the *compression* travel.

6.6 Rebound — The relative displacement of the *sprung* and *unsprung masses* in a suspension system in which the distance between the masses increases from that at static condition.

6.6.1 REBOUND CLEARANCE — The maximum displacement in *rebound* of the *sprung mass* relative to the *wheel center* permitted by the suspension system, from the normal load position.

6.6.2 METAL-TO-METAL POSITION (REBOUND) — The point of maximum *rebound* travel limited by interference of substantially rigid members.

6.6.3 REBOUND STOP — An elastic member which increases the *wheel rate* toward the end of the *rebound* travel.

The rebound stop may also act to limit the *rebound* travel.

6.7 Center of Parallel Wheel Motion — The center of curvature of the path along which each of a pair of *wheel centers* moves in a longitudinal vertical plane relative to the *sprung mass* when both wheels are equally displaced.

6.8 Torque Arm

6.8.1 TORQUE-ARM CENTER IN BRAKING — The instantaneous center in a vertical longitudinal plane through the *wheel center* about which the wheel moves relative to the *sprung mass* when the brake is locked.

6.8.2 TORQUE-ARM CENTER IN DRIVE — The instantaneous center in a vertical longitudinal plane through the *wheel center* about which the wheel moves relative to the *sprung mass* when the drive mechanism is locked at the power source.

6.8.3 TORQUE-ARM RADIUS — The horizontal distance from the *torque-arm center* to the *wheel center.*

7. Tires and Wheels

7.1 General Nomenclature

7.1.1 STANDARD LOADS AND INFLATIONS — Those combinations of loads and inflations up to the maximum load and inflation recommended by the Tire & Rim Association and published in the yearly editions of the Tire & Rim Association Yearbook.

7.1.2 RIM DIAMETER — The diameter at the intersection of the *bead* seat and the flange. (See Tire & Rim Association Yearbook.) Nominal rim diameter (i.e., 14, 15, 16.5, etc.) is commonly used.

7.1.3 RIM WIDTH — The distance between the inside surfaces or the rim flanges. (See Tire & Rim Association Yearbook.)

7.1.4 TIRE SECTION WIDTH — The width of the unloaded new tire mounted on specified rim, inflated to the normal recommended pressure, including the normal sidewalls but not including protective rib, bars, and decorations. (See Tire & Rim Association Yearbook.)

7.1.5 TIRE OVERALL WIDTH — The width of the unloaded new tire, mounted on specified rim, inflated to the normal recommended pressure, including protective rib, bars, and decorations. (See Tire & Rim Association Yearbook.)

7.1.6 TIRE SECTION HEIGHT — Half the difference between the tire *outside diameter* and the nominal *rim diameter.*

7.1.7 OUTSIDE DIAMETER — The maximum diameter of the new unloaded tire inflated to the normal recommended pressure and mounted on a specified rim. (See Airplane Section, Tire & Rim Association Yearbook.)

7.1.8 FLAT TIRE RADIUS — The distance from the *spin axis* to the road surface of a loaded tire on a specified rim at zero inflation.

7.1.9 DEFLECTION (STATIC) — The radial difference between the undeflected tire radius and the static loaded radius, under specified loads and inflation.

7.1.9.1 Percent Deflection — The static deflection expressed as a percentage of the unloaded section height above the top of the rim flange.

7.1.10 TIRE RATE (STATIC) — See 4.3.2.

7.1.11 SIDEWALL — The portion of either side of the tire which connects the *bead* with the *tread.*

7.1.11.1 Sidewall Rib — A raised circumferential rib located on the *sidewall.*

7.1.12 BEAD — The portion of the tire which fits onto the rim of the wheel.

7.1.12.1 Bead Base — The approximately cylindrical portion of the *bead* that forms its inside diameter.

7.1.12.2 Bead Toe — That portion of the *bead* which joins the *bead base* and the inside surface of the tire.

7.1.13 TREAD (TIRE) — The peripheral portion of the tire, the exterior of which is designed to contact the road surface.

7.1.13.1 Tread Contour — The cross-sectional shape of *tread* surface of an inflated unloaded tire neglecting the *tread pattern* depressions.

7.1.13.2 Tread Radius — The radius or combination of radii describing the *tread contour.*

7.1.13.3 Tread Arc Width — The distance measured along the *tread contour* of an unloaded tire between one edge of the *tread* and the other. For tires with rounded tread edges, the point of measurement is that point in space which is at the intersection of the *tread radius* extended until it meets the prolongation of the upper sidewall contour.

7.1.13.4 Tread Chord Width — The distance measured parallel to the *spin axis* of an unloaded tire between one edge of the *tread* and the other. For tires with rounded tread edges, the point of measurement is that point in space which is at the intersection of the *tread radius* extended until it meets the prolongation of the upper sidewall contour.

7.1.13.5 Tread Contact Width — The distance between the extreme edges of road contact at a specified load and pressure measured parallel to the Y' axis at zero *slip angle* and zero *inclination angle.*

7.1.13.6 Tread Contact Length — The perpendicular distance between the tangent to edges of the leading and following points of road contact and parallel to the *wheel plane.*

7.1.13.7 Tread Depth — The distance between the base of a tire *tread* groove and a line tangent to the surface of the two adjacent *tread* ribs or rows.

7.1.13.8 Gross Contact Area — The total area enclosing the pattern of the tire *tread* in contact with a flat surface, including the area of grooves or voids.

7.1.13.9 Net Contact Area — The area enclosing the pattern of the tire *tread* in contact with a flat surface, excluding the area of grooves or other depressions.

7.1.13.10 Tread Pattern — The molded configuration on the face of the *tread*. It is generally composed of ribs, rows, grooves, bars, lugs, and the like.

7.2 Rolling Characteristics

7.2.1 LOADED RADIUS (R_ℓ) — The distance from the *center of tire contact* to the *wheel center* measured in the wheel plane.

7.2.2 STATIC LOADED RADIUS — The *loaded radius* of a stationary tire inflated to normal recommended pressure.

NOTE: In general, static loaded radius is different from the radius of slowly rolling tire. Static radius of a tire rolled into position may be different from that of the tire loaded without being rolled.

7.2.3 SPIN AXIS — The axis of rotation of the wheel. (See Fig. 1.)

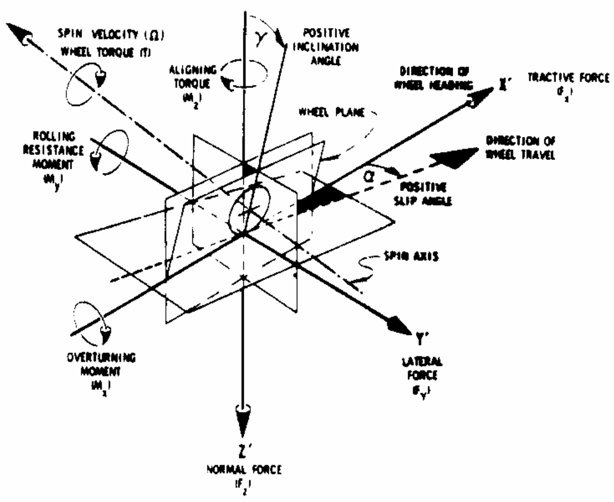

FIG. 1 — TIRE AXIS SYSTEM

7.2.4 SPIN VELOCITY (Ω) — The angular velocity of the wheel on which the tire is mounted, about its *spin axis*. Positive spin velocity is shown in Fig. 1.

7.2.5 FREE-ROLLING TIRE — A loaded rolling tire operated without application of *driving* or *braking torque*.

7.2.6 STRAIGHT FREE-ROLLING TIRE — A *free-rolling tire* moving in a straight line at zero *inclination angle* and zero *slip angle*.

7.2.7 LONGITUDINAL SLIP VELOCITY — The difference between the *spin velocity* of the driven or braked tire and the *spin velocity* of the *straight free-rolling tire*. Both spin velocities are measured at the same linear velocity at the wheel center in the X' direction. A positive value results from *driving torque*.

7.2.8 LONGITUDINAL SLIP (PERCENT SLIP) — The ratio of the *longitudinal slip velocity* to the *spin velocity* of the *straight free-rolling tire* expressed as a percentage.

NOTE: This quantity should not be confused with the slip number that frequently appears in kinematic analysis of tires in which the *spin velocity* appears in the denominator.

7.2.9 EFFECTIVE ROLLING RADIUS (R_e) — The ratio of the linear velocity of the wheel center in the X' direction to the *spin velocity*. (See 7.3.1.)

7.2.10 WHEEL SKID — The occurrence of sliding between the tire and road interface which takes place within the entire *contact area*. Skid can result from braking, driving and/or cornering.

7.3 Tire Forces and Moments

7.3.1 TIRE AXIS SYSTEM (FIG. 1) — The origin of the tire axis system is the *center of tire contact*. The X' axis is the intersection of the wheel plane and the road plane with a positive direction forward. The Z' axis is perpendicular to the road plane with a positive direction downward. The Y' axis is in the road plane, its direction being chosen to make the axis system orthogonal and right-hand.

7.3.2 TIRE ANGLES

7.3.2.1 Slip Angle (α) — The angle between the X' axis and direction of travel of the *center of tire contact*.

7.3.2.2 Inclination Angle (γ) — The angle between the Z' axis and the *wheel plane*.

7.3.3 TIRE FORCES — The external force acting on the tire by the road having the following components:

7.3.3.1 Longitudinal Force (F_x) — The component of the *tire force* vector in the X' direction.

7.3.3.2 Driving Force — The *longitudinal force* resulting from *driving torque* application.

7.3.3.3 Driving Force Coefficient — The ratio of the *driving force* to the *vertical load.*

7.3.3.4 Braking Force — The negative *longitudinal force* resulting from *braking torque* application.

7.3.3.5 Braking Force Coefficient (Braking Coefficient) — The ratio of the *braking force* to the *vertical load.*

7.3.3.6 Rolling Resistance Force — The negative *longitudinal force* resulting from energy losses due to deformations of a rolling tire.

NOTE: This force can be computed from the forces and moments acting on the tire by the road.

$$F_r = (M_y \cos\gamma + M_z \sin\gamma)/R_\ell$$

7.3.3.7 Rolling Resistance Force Coefficient (Coefficient of Rolling Resistance) — The ratio of the *rolling resistance* to the *vertical load.*

7.3.3.8 Lateral Force (F_y) — The component of the *tire force* vector in the Y' direction.

7.3.3.9 Lateral Force Coefficient — The ratio of the *lateral force* to the *vertical load.*

7.3.3.10 Slip Angle Force — The *lateral force* when the *inclination angle* is zero and *plysteer* and *conicity* forces have been subtracted.

7.3.3.11 Camber Force (Camber Thrust) — The *lateral force* when the *slip angle* is zero and the *plysteer* and *conicity* forces have been subtracted.

7.3.3.12 Normal Force (F_z) — The component of the *tire force* vector in the Z' direction.

7.3.3.13 Vertical Load — The normal reaction of the tire on the road which is equal to the negative of the normal force.

7.3.3.14 Central Force — The component of the *tire force* vector in the direction perpendicular to the direction of travel of the *center of tire contact.* Central force is equal to *lateral force* times cosine of *slip angle* minus *longitudinal force* times sine of *slip angle.*

7.3.3.15 Tractive Force — The component of the *tire force* vector in the direction of travel of the *center of tire contact.* Tractive force is equal to *lateral force* times sine of *slip angle* plus *longitudinal force* times cosine of *slip angle.*

7.3.3.16 Drag Force — The negative *tractive force.*

7.3.4 TIRE MOMENTS — The external moments acting on the tire by the road having the following components:

7.3.4.1 Overturning Moment (M_x) — The component of the *tire moment* vector tending to rotate the tire about the X' axis, positive clockwise when looking in the positive direction of the X' axis.

7.3.4.2 Rolling Resistance Moment (M_y) — The component of the *tire moment* vector tending to rotate the tire about the Y' axis, positive clockwise when looking in the positive direction of the Y' axis.

7.3.4.3 Aligning Torque (Aligning Moment) (M_z) — The component of the *tire moment* vector tending to rotate the tire about the Z' axis, positive clockwise when looking in the positive direction of the Z' axis.

7.3.4.4 Wheel Torque (T) — The external torque applied to the tire from the vehicle about the *spin axis*; positive *wheel torque* is shown in Fig. 1.

7.3.4.5 Driving Torque — The positive *wheel torque.*

7.3.4.6 Braking Torque — The negative *wheel torque.*

7.4 Tire Force and Moment Stiffness — (may be evaluated at any set of operating conditions)

7.4.1 CORNERING STIFFNESS — The negative of the rate of change of *lateral force* with respect to change in *slip angle,* usually evaluated at zero *slip angle.*

7.4.2 CAMBER STIFFNESS — The rate of change of *lateral force* with respect to change in *inclination angle,* usually evaluated at zero *inclination angle.*

7.4.3 BRAKING (DRIVING) STIFFNESS — The rate of change of *longitudinal force* with respect to change in *longitudinal slip*, usually evaluated at zero *longitudinal slip.*

7.4.4 ALIGNING STIFFNESS (ALIGNING TORQUE STIFFNESS) — The rate of change of *aligning* torque with respect to change in *slip angle*, usually evaluated at zero *slip angle.*

7.5 Normalized Tire Force and Moment Stiffnesses (Coefficients)

7.5.1 CORNERING STIFFNESS COEFFICIENT (CORNERING COEFFICIENT) — The ratio of *cornering stiffness* of a *straight free-rolling tire* to the *vertical load.*

NOTE: Although the term cornering coefficient has been used in a number of technical papers, for consistency with definitions of other terms using the word coefficient, the term cornering stiffness coefficient is preferred.

7.5.2 CAMBER STIFFNESS COEFFICIENT (CAMBER COEFFICIENT) — The ratio of camber stiffness of a *straight free-rolling tire* to the *vertical load.*

7.5.3 BRAKING (DRIVING) STIFFNESS COEFFICIENT — The ratio of braking (driving) stiffness of a *straight free-rolling tire* to the *vertical load.*

7.5.4 ALIGNING STIFFNESS COEFFICIENT (ALIGNING TORQUE COEFFICIENT) — The ratio of *aligning stiffness* of a *straight free-rolling tire* to the *vertical load.*

7.6 Tire Traction Coefficients

7.6.1 LATERAL TRACTION COEFFICIENT — The maximum value of *lateral force coefficient* which can be reached on a *free-rolling tire* for a given road surface, environment and operating condition.

7.6.2 DRIVING TRACTION COEFFICIENT — The maximum value of *driving force coefficient* which can be reached on a given tire and road surface for a given environment and operating condition.

7.6.3 BRAKING TRACTION COEFFICIENT — The maximum of the *braking force coefficient* which can be reached without locking a wheel on a given tire and road surface for a given environment and operating condition.

7.6.3.1 Sliding Braking Traction Coefficient — The value of the *braking force coefficient* of a tire obtained on a locked wheel on a given tire and road surface for a given environment and operating condition.

7.7 Tire Associated Noise and Vibrations

7.7.1 TREAD NOISE — Airborne sound (up to 5000 Hz) except squeal and slap produced by the interaction between the tire and the road surface.

7.7.1.1 Sizzle — A *tread noise* (up to 4000 Hz) characterized by a soft frying sound, particularly noticeable on a very smooth road surface.

7.7.2 SQUEAL — Narrow band airborne tire noise (150-800 Hz) resulting from either *longitudinal slip* or *slip angle* or both.

7.7.2.1 Cornering Squeal — The *squeal* produced by a *free-rolling tire* resulting from *slip angle.*

7.7.2.2 Braking (Driving) Squeal — The *squeal* resulting from *longitudinal slip.*

7.7.3 THUMP — A periodic vibration and/or audible sound generated by the tire and producing a pounding sensation which is synchronous with wheel rotation.

7.7.4 ROUGHNESS — Vibration (15-100 Hz) perceived tactually and/or audibly, generated by a rolling tire on a smooth road surface and producing the sensation of driving on a coarse or irregular surface.

7.7.5 HARSHNESS — Vibrations (15-100 Hz) perceived tactually and/or audibly, produced by interaction of the tire with road irregularities.

7.7.6 SLAP — Airborne smacking noise produced by a tire traversing road seams such as tar strips and expansion joints.

7.8 Tire and Wheel Non-Uniformity Characteristics

7.8.1 RADIAL RUN-OUT

7.8.1.1 Peak-to-Peak Radial Wheel Run-Out — The difference between the maximum and minimum values of the wheel *bead* seat radius, measured in a plane perpendicular to the *spin axis* (measured separately for each *bead* seat).

7.8.1.2 Peak-to-Peak Unloaded Radial Tire Run-Out — The difference between maximum and minimum undeflected values of the tire radius, measured in plane perpendicular to the *spin axis* on a true running wheel.

7.8.1.3 Peak-to-Peak Loaded Radial Tire Run-Out — The difference between maximum and minimum values of the *loaded radius* on a true running wheel.

7.8.2 LATERAL RUN-OUT

7.8.2.1 Peak-to-Peak Lateral Wheel Run-Out — The difference between maximum and minimum indicator readings, measured parallel to the *spin axis* on the inside vertical portion of a rim flange (measured separately for each flange).

7.8.2.2 Peak-to-Peak Lateral Tire Run-Out — The difference between maximum and minimum indicator readings, measured parallel to the *spin axis* at the point of maximum *tire section,* on a true running wheel (measured separately for each sidewall).

7.8.3 RADIAL FORCE VARIATION — The periodic variation of the *normal force* of a loaded *straight free-rolling tire* which repeats each revolution at a fixed *loaded radius*, given mean *normal force*, constant speed, given inflation pressure and test surface curvature.

7.8.3.1 Peak-to-Peak (Total) Radial Force Variation — The difference between maximum and minimum values of the *normal force* during one revolution of the tire.

7.8.3.2 First Order Radial Force Variation — The peak-to-peak amplitude of the fundamental frequency component of the Fourier series representing *radial force variation.* Its frequency is equal to the rotational frequency of the tire.

7.8.4 LATERAL FORCE VARIATION — The periodic variation of lateral force of a *straight free-rolling tire* which repeats each revolution, at a fixed *loaded radius,* given mean *normal force*, constant speed, given inflation pressure and test surface curvature.

7.8.4.1 Peak-to-Peak (Total) Lateral Force Variation — The difference between the maximum and minimum values of the *lateral force* during one revolution of the tire.

7.8.4.2 First Order Lateral Force Variation — The peak-to-peak amplitude of the fundamental frequency component of the Fourier series representing *lateral force variation.* Its frequency is equal to the rotational frequency of the tire.

7.8.5 LATERAL FORCE OFFSET — The average *lateral force* of a *straight free-rolling tire.*

7.8.5.1 Ply Steer Force — The component of *lateral force offset* which does not change sign (with respect to the *Tire Axis System*) with a change in direction of rotation (positive along positive Y' axis). The force remains positive when it is directed away from the serial number on the right side tire and toward the serial number on the left side tire.

7.8.5.2 Conicity Force — The component of *lateral force offset* which changes sign (with respect to the *Tire Axis System*) with a change in direction of rotation (positive away from the serial number or toward the whitewall). The force is positive when it is directed away from the serial number on the right side tire and negative when it is directed toward the serial number on the left side tire.

8. *Kinematics — Force and Moments Notation*

8.1 Earth-Fixed Axis System (X,Y,Z) — This system is a right-hand orthogonal axis system fixed on the earth. The trajectory of the vehicle is described with respect to this earth-fixed axis system. The X- and Y-axis are in a horizontal plane and the Z-axis is directed downward.

8.2 Vehicle Axis System (x,y,z) — This system is a right-hand orthogonal axis system fixed in a vehicle such that with the vehicle moving steadily in a straight line on a level road, the x-axis is substantially horizontal, points forward, and is in the longitudinal plane of symmetry. The y-axis points to driver's right and the z-axis points downward. (See Fig. 2.)

8.3 Angular Orientation — The orientation of the *vehicle axis system (x,y,z)* with respect to the *earth-fixed axis system (X,Y,Z)* is given by a sequence of three angular rotations. The following sequence of rotations (see Note 6), starting from a condition in which the two sets of axes are initially aligned, is defined to be the standard:

(1) A yaw rotation, ψ, about the aligned z- and Z-axis.
(2) A pitch rotation, Θ, about the vehicle y-axis.
(3) A roll rotation, ϕ, about the vehicle x-axis.

8.4 Motion Variables

8.4.1 VEHICLE VELOCITY — The vector quantity expressing velocity of a point in the vehicle relative to the *earth-fixed axis system (X,Y,Z).* The following motion variables are components of this vector resolved with respect to the moving *vehicle axis system (x,y,z).*

8.4.1.1 Longitudinal Velocity (u) of a point in the vehicle is the component of the vector velocity in the x-direction.

8.4.1.2 Side Velocity (v) of a point in the vehicle is the component of the vector velocity in the y-direction.

8.4.1.3 Normal Velocity (w) of a point in the vehicle is the component of the vector velocity in the z-direction.

8.4.1.4 Forward Velocity of a point in the vehicle is the component of the vector velocity perpendicular to the y-axis and parallel to the road plane.

8.4.1.5 Lateral Velocity of a point in the vehicle is the component of the vector velocity perpendicular to the x-axis and parallel to the road plane.

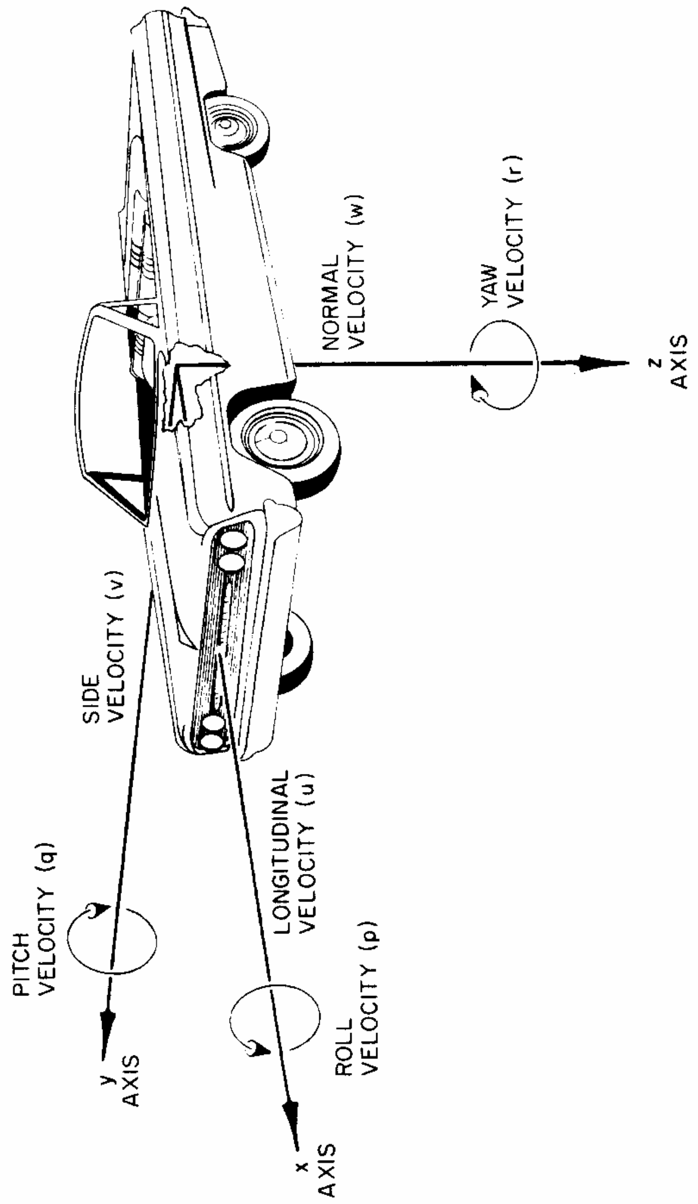

FIG. 2 — DIRECTIONAL CONTROL AXIS SYSTEM

8.4.1.6 Roll Velocity (p) — The angular velocity about the x-axis.

8.4.1.7 Pitch Velocity (q) — The angular velocity about the y-axis.

8.4.1.8 Yaw Velocity (r) — The angular velocity about the z-axis.

8.4.2 VEHICLE ACCELERATION — The vector quantity expressing the acceleration of a point in the vehicle relative to the *earth-fixed axis system (X,Y,Z)*. The following motion variables are components of this vector, resolved with respect to the moving *vehicle axis system.*

8.4.2.1 Longitudinal Acceleration — The component of the vector acceleration of a point in the vehicle in the x-direction.

8.4.2.2 Side Acceleration — The component of the vector acceleration of a point in the vehicle in the y-direction.

8.4.2.3 Normal Acceleration — The component of the vector acceleration of a point in the vehicle in the z-direction.

8.4.2.4 Lateral Acceleration — The component of the vector acceleration of a point in the vehicle perpendicular to the vehicle x-axis and parallel to the road plane. (See Note 7.)

8.4.2.5 Centripetal Acceleration — The component of the vector acceleration of a point in the vehicle perpendicular to the tangent to the path of that point and parallel to the road plane.

8.4.3 HEADING ANGLE (ψ) — The angle between the trace on the X-Y plane of the vehicle x-axis and the X-axis of the *earth-fixed axis system.* (See Fig. 3.)

8.4.4 SIDESLIP ANGLE (ATTITUDE ANGLE) (β) — The angle between the traces on the X-Y plane of the vehicle x-axis and the vehicle velocity vector at some specified point in the vehicle. Sideslip angle is shown in Fig. 3 as a negative angle.

8.4.5 SIDESLIP ANGLE GRADIENT — The rate of change of *sideslip angle* with respect to change in steady-state *lateral acceleration* on a level road at a given *trim* and test conditions.

8.4.6 COURSE ANGLE (υ) — The angle between the trace of the vehicle velocity vector on the X-Y plane and X-axis of the *earth-fixed axis system.* A positive course angle is shown in Fig. 3. Course angle is the sum of *heading angle* and *sideslip angle* ($\upsilon = \psi + \beta$).

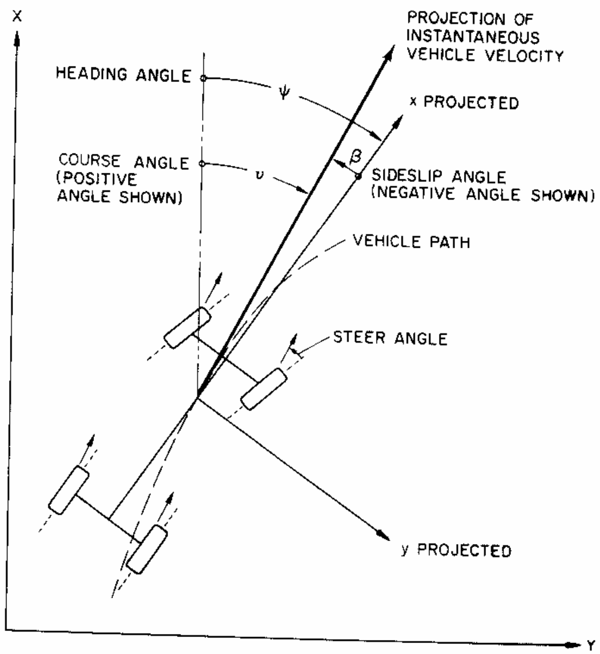

FIG. 3 — HEADING, SIDESLIP, AND COURSE ANGLES

8.4.7 VEHICLE ROLL ANGLE — The angle between the vehicle y-axis and the ground plane.

8.4.8 VEHICLE ROLL GRADIENT — The rate of change in *vehicle roll angle* with respect to change in steady-state *lateral acceleration* on a level road at a given *trim* and test conditions.

8.4.9 VEHICLE PITCH ANGLE — The angle between the vehicle x-axis and the ground plane.

8.5 Forces — The external forces acting on the vehicle can be summed into one force vector having the following components:

8.5.1 LONGITUDINAL FORCE (F_x) — The component of the force vector in the x-direction.

8.5.2 SIDE FORCE (F_y) — The component of the force vector in the y-direction.

8.5.3 NORMAL FORCE (F_z) — The component of the force vector in the z-direction.

8.6 Moments — The external moments acting on the vehicle can be summed into one moment vector having the following components:

8.6.1 ROLLING MOMENT (M_x) — The component of the moment vector tending to rotate the vehicle about the x-axis, positive clockwise when looking in the positive direction of the x-axis.

8.6.2 PITCHING MOMENT (M_y) — The component of the moment vector tending to rotate the vehicle about the y-axis, positive clockwise when looking in the positive direction of the y-axis.

8.6.3 YAWING MOMENT (M_z) — The component of the moment vector tending to rotate the vehicle about the z-axis, positive clockwise when looking in the positive direction of the z-axis.

9. *Directional Dynamics*

9.1 Control Modes

9.1.1 POSITION CONTROL — That mode of vehicle control wherein inputs or restraints are placed upon the steering system in the form of displacements at some control point in the steering system (front wheels, Pitman arm, steering wheel), independent of the force required.

9.1.2 FIXED CONTROL — That mode of vehicle control wherein the position of some point in the steering system (front wheels, Pitman arm, steering wheel) is held fixed. This is a special case of *position control.*

9.1.3 FORCE CONTROL — That mode of vehicle control wherein inputs or restraints are placed upon the steering system in the form of forces, independent of the displacement required.

9.1.4 FREE CONTROL — That mode of vehicle control wherein no restraints are placed upon the steering system. This is a special case of *force control.*

9.2 Vehicle Response — The vehicle motion resulting from some internal or external input to the vehicle. Response tests can be used to determine the *stability* and control characteristics of a vehicle.

9.2.1 STEERING RESPONSE — The vehicle motion resulting from an input to the steering (control) element. (See Note 8.)

9.2.2 DISTURBANCE RESPONSE — The vehicle motion resulting from unwanted force or displacement inputs applied to the vehicle. Examples of disturbances are wind forces or vertical road displacements.

9.2.3 STEADY-STATE — Steady-state exists when periodic (or constant) vehicle responses to periodic (or constant) control and/or disturbance inputs do not change over an arbitrarily long time. The motion responses in steady-state are referred to as steady-state responses. This definition does not require the vehicle to be operating in a straight line or on a level road surface. It can also be in a turn of constant radius or on a cambered road surface.

9.2.4 TRANSIENT STATE — Transient state exists when the motion responses, the external forces relative to the vehicle, or the control positions are changing with time. (See Note 9.)

9.2.5 TRIM — The *steady-state* (that is, equilibrium) condition of the vehicle with constant input which is used as the reference point for analysis of dynamic vehicle *stability* and control characteristics.

9.2.6 STEADY-STATE RESPONSE GAIN — The ratio of change in the *steady-state* response of any motion variable with respect to change in input at a given *trim*.

9.2.7 STEERING SENSITIVITY (CONTROL GAIN) — The change in *steady-state lateral acceleration* on a level road with respect to change in *steering wheel angle* at a given *trim* and test conditions.

9.3 Stability — (See Note 10.)

9.3.1 ASYMPTOTIC STABILITY — Asymptotic stability exists at a prescribed *trim* if, for any small temporary change in disturbance or *control input*, the vehicle will approach the motion defined by the *trim*.

9.3.2 NEUTRAL STABILITY — Neutral stability exists at a prescribed *trim* if, for any small temporary change in disturbance or *control input*, the resulting motion of the vehicle remains close to, but does not return to, the motion defined by the *trim*.

9.3.3 DIVERGENT INSTABILITY — Divergent instability exists at a prescribed *trim* if any small temporary disturbance or *control input* causes an ever-increasing *vehicle response* without *oscillation*. (See Note 11.)

9.3.4 OSCILLATORY INSTABILITY — Oscillatory instability exists if a small temporary disturbance or *control input* causes an oscillatory *vehicle response* of ever-increasing amplitude about the initial *trim*. (See Note 12.)

9.4 Suspension Steer and Roll Properties — (Fig. 4) (See Note 13.)

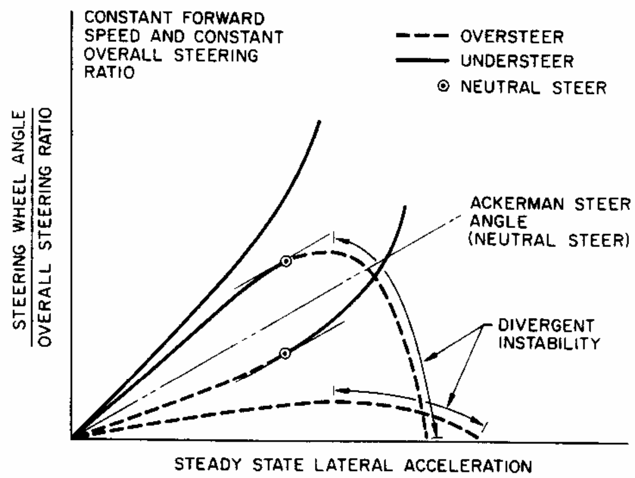

FIG. 4 — STEER PROPERTIES (SEE NOTE 17)

9.4.1 STEER ANGLE (δ) — The angle between the projection of a longitudinal axis of the vehicle and the line of intersection of the *wheel plane* and the road surface. Positive angle is shown in Fig. 3.

9.4.2 ACKERMAN STEER ANGLE (δ_a) — The angle whose tangent is the wheelbase divided by the radius of turn.

9.4.3 ACKERMAN STEER ANGLE GRADIENT — The rate of change of *Ackerman steer angle* with respect to change in *steady-state lateral acceleration* on a level road at a given *trim* and test conditions. (See Note 14.)

9.4.4 STEERING WHEEL ANGLE — Angular displacement of the steering wheel measured from the straight-ahead position (position corresponding to zero average *steer angle* of a pair of steered wheels).

9.4.5 STEERING WHEEL ANGLE GRADIENT — The rate of change in the *steering wheel angle* with respect to change in *steady-state lateral acceleration* on a level road at a given *trim* and test conditions.

9.4.6 OVERALL STEERING RATIO — The rate of change of *steering wheel angle* at a given steering wheel *trim* position, with respect to change in average *steer angle* of a pair of steered wheels, assuming an infinitely stiff steering system with no roll of the vehicle (see Note 15).

9.4.7 UNDERSTEER/OVERSTEER GRADIENT — The quantity obtained by subtracting the *Ackerman steer angle gradient* from the ratio of the *steering wheel angle gradient* to the *overall steering ratio.*

9.4.8 NEUTRAL STEER — A vehicle is neutral steer at a given *trim* if the ratio of the *steering wheel angle gradient* to the *overall steering ratio* equals the *Ackerman steer angle gradient.*

9.4.9 UNDERSTEER — A vehicle is understeer at a given *trim* if the ratio of the *steering wheel angle gradient* to the *overall steering ratio* is greater than the *Ackerman steer angle gradient.*

9.4.10 OVERSTEER — A vehicle is oversteer at a given *trim* if the ratio of the *steering wheel angle gradient* to the *overall steering ratio* is less than the *Ackerman steer angle gradient.*

9.4.11 STEERING WHEEL TORQUE — The torque applied to the steering wheel about its axis of rotation.

9.4.12 STEERING WHEEL TORQUE GRADIENT — The rate of change in the *steering wheel torque* with respect to change in *steady-state lateral acceleration* on a level road at a given *trim* and test conditions.

9.4.13 CHARACTERISTIC SPEED — That forward speed for an understeer vehicle at which the steering sensitivity at zero *lateral acceleration* trim is one-half the steering sensitivity of a *neutral steer* vehicle.

9.4.14 CRITICAL SPEED — That forward speed for an *oversteer* vehicle at which the steering sensitivity at zero *lateral acceleration trim* is infinite.

9.4.15 NEUTRAL STEER LINE — The set of points in the x-z plane at which external lateral forces applied to the *sprung mass* produce no *steady-state yaw velocity.*

9.4.16 STATIC MARGIN — The horizontal distance from the center of gravity to the *neutral steer line* divided by the wheelbase. It is positive if the center of gravity is forward of the *neutral steer line.*

9.4.17 SUSPENSION ROLL — The rotation of the vehicle *sprung mass* about the x-axis with respect to a transverse axis joining a pair of *wheel centers.*

9.4.18 SUSPENSION ROLL ANGLE — The angular displacement produced by *suspension roll.*

9.4.19 SUSPENSION ROLL GRADIENT — The rate of change in the *suspension roll angle* with respect to change in *steady-state lateral acceleration* on a level road at a given *trim* and test conditions.

9.4.20 ROLL STEER — The change in *steer angle* of front or rear wheels due to *suspension roll.*

9.4.20.1 Roll Understeer — *Roll steer* which increases vehicle *understeer* or decreases vehicle *oversteer.*

9.4.20.2 Roll Oversteer — *Roll steer* which decreases vehicle *understeer* or increases vehicle *oversteer.*

9.4.21 ROLL STEER COEFFICIENT — The rate of change in *roll steer* with respect to change in *suspension roll angle* at a given *trim.*

9.4.22 COMPLIANCE STEER — The change in *steer angle* of front or rear wheels resulting from compliance in suspension and steering linkages and produced by forces and/or moments applied at the tire-road contact.

9.4.22.1 Compliance Understeer — *Compliance steer* which increases vehicle *understeer* or decreases vehicle *oversteer.*

9.4.22.2 Compliance Oversteer — *Compliance steer* which decreases vehicle *understeer* or increases vehicle *oversteer.*

9.4.23 COMPLIANCE STEER COEFFICIENT — The rate of change in *compliance steer* with respect to change in forces or moments applied at the tire-road contact.

9.4.24 ROLL CAMBER — The camber displacements of a wheel resulting from *suspension roll.*

9.4.25 ROLL CAMBER COEFFICIENT — The rate of change in wheel *inclination angle* with respect to change in *suspension roll angle.*

9.4.26 COMPLIANCE CAMBER — The camber motion of a wheel resulting from compliance in suspension linkages and produced by forces and/or moments applied at the tire-road contact.

9.4.27 COMPLIANCE CAMBER COEFFICIENT — The rate of change in wheel *inclination angle* with respect to change in forces or moments applied at the tire-road contact.

9.4.28 ROLL CENTER — The point in the transverse vertical plane through any pair of *wheel centers* at which lateral forces may be applied to the *sprung mass* without producing *suspension roll.* (See Note 16.)

9.4.29 ROLL AXIS — The line joining the front and rear *roll centers.*

9.4.30 SUSPENSION ROLL STIFFNESS — The rate of change in the restoring couple exerted by the suspension of a pair of wheels on the *sprung mass* of the vehicle with respect to change in *suspension roll angle.*

9.4.31 VEHICLE ROLL STIFFNESS — Sum of the separate *suspension roll stiffnesses.*

9.4.32 ROLL STIFFNESS DISTRIBUTION — The distribution of the *vehicle roll stiffness* between front and rear suspension expressed as percentage of the *vehicle roll stiffness.*

9.5 Tire Load Transfer

9.5.1 TIRE LATERAL LOAD TRANSFER — The *vertical load* transfer from one of the front tires (or rear tires) to the other that is due to acceleration, rotational, or inertial effects in the lateral direction.

9.5.2 TIRE LATERAL LOAD TRANSFER DISTRIBUTION — The distribution of the total *tire lateral load transfer* between front and rear tires expressed as the percentage of the total.

9.5.3 TIRE LONGITUDINAL LOAD TRANSFER — The *vertical load* transferred from a front tire to the corresponding rear tire that is due to acceleration, rotational, or inertial effects in the longitudinal direction.

9.5.4 OVERTURNING COUPLE — The overturning moment on the vehicle with respect to a central, longitudinal axis in the road plane due to lateral acceleration and roll acceleration.

9.5.5 OVERTURNING COUPLE DISTRIBUTION — The distribution of the total *overturning couple* between the front and rear suspensions expressed as the percentage of the total.

10. Aerodynamic Nomenclature

10.1 Aerodynamic Motion Variables

10.1.1 AMBIENT WIND VELOCITY (v_a) — The horizontal component of the air mass velocity relative to the *earth-fixed axis system* in the vicinity of the vehicle.

10.1.2 AMBIENT WIND ANGLE (v_a) — The angle between the X axis of the *earth-fixed axis system* and the *ambient wind velocity* vector. A positive ambient wind angle is shown in Fig. 5.

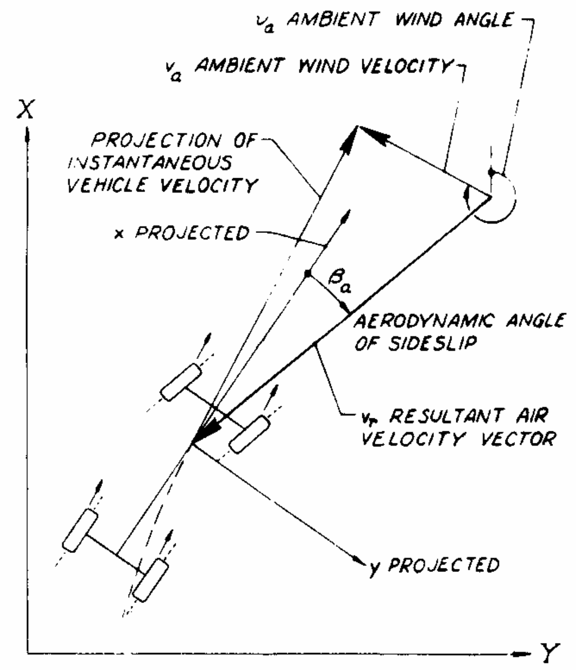

FIG. 5 — WIND VECTORS

10.1.3 RESULTANT AIR VELOCITY VECTOR (v_r) — The vector difference of the *ambient wind velocity* vector and the projection of the velocity vector of the vehicle on the X-Y plane.

10.1.4 AERODYNAMIC SIDESLIP ANGLE (β_a) — The angle between the traces on the vehicle x-y plane of the vehicle x-axis and the resultant air velocity vector at some specified point in the vehicle.

10.1.5 AERODYNAMIC ANGLE OF ATTACK (α_a) — The angle between the vehicle x-axis and the trace of the resultant air velocity vector on a vertical plane containing the vehicle x-axis.

10.2 Aerodynamic Force and Moment Coefficients

10.2.1 REFERENCE DIMENSIONS

10.2.1.1 Vehicle Area (A) — The projected frontal area including tires and underbody parts.

10.2.1.2 Vehicle Wheelbase (ℓ) — The characteristic length upon which aerodynamic moment coefficients are based.

10.2.2 STANDARD AIR PROPERTIES

10.2.2.1 The density of standard dry air shall be taken as 2378 x 10^{-6} slugs/ft at 59°F and 29.92 in Hg.

10.2.2.2 The viscosity of standard dry air shall be taken as 373 x 10^{-9} slugs/ft-sec.

10.2.3 FORCE COEFFICIENTS

10.2.3.1 The Longitudinal Force Coefficient (C_x) is based on the aerodynamic force acting on the vehicle in the x-direction (as established by 8.2) and is defined as:

$$C_x = F_x/(qA)$$

where q is dynamic pressure at any given relative air velocity as given by the formula

$$q = (pv^2)/2$$

10.2.3.2 Side Force Coefficient (C_y) is based on the aerodynamic force acting on the vehicle in the y-direction (as established by 8.2) and is defined as:

$$C_y = F_y/(qA)$$

where q is the dynamic pressure at any given relative air velocity as given by the formula

$$q = (pv^2)/2$$

10.2.3.3 The Normal Force Coefficient (C_z) is based on the aerodynamic force acting in the z-direction (as established by 8.2) and is defined as:

$$C_z = F_z/(qA)$$

10.2.4 MOMENT COEFFICIENTS

10.2.4.1 The Rolling Moment Coefficient (C_{Mx}) is based on the rolling moment deriving from the distribution of aerodynamic forces acting on the vehicle and is defined as:

$$C_{Mx} = M_x/(qA\ell)$$

10.2.4.2 The Pitching Moment Coefficient (C_{My}) is based on the pitching moment deriving from the distribution of aerodynamic forces acting on the vehicle and is defined as:

$$C_{My} = M_y/(qA\ell)$$

10.2.4.3 The Yawing Moment Coefficient (C_{Mz}) is based on the yawing moment deriving from the distribution of aerodynamic forces acting on the vehicle and is defined as:

$$C_{Mz} = M_z/(qA\ell)$$

NOTES

1. The *center of tire contact* may not be the geometric center of the tire contact area due to distortion of the tire produced by applied forces.

2. The *static toe (inches)* is equal to the sum of the toe angles (degrees) of the left and right wheels multiplied by the ratio of tire diameter (inches) to 57.3.

   If the toe angles on the left and right wheels are the same and the outside diameter of tire is 28.65 in (727.7 mm), the *static toe (inches)* is equal to *static toe angle (degrees).*

3. It is important to recognize that to make axis transformations and resolve these forces with respect to the direction of vehicle motion, it is essential to measure all six force and moment components defined in 7.3.3.1-7.3.3.6 and 7.3.4.1-7.3.4.6.

4. This *rolling resistance force* definition has been generalized so that it applies to wheels which are driven or braked. The *wheel torque* can be expressed in terms of the *longitudinal force*, *rolling resistance force*, and *loaded radius* by the equation

$$T = (F_x + F_r)R_\ell$$

For a *free-rolling wheel,* the *rolling resistance force* is therefore the negative of the *longitudinal force.*

5. For small *slip* and *inclination angles*, the *lateral force* developed by the tire can be approximated by

$$F_y = -C_\alpha \alpha + C_\gamma \gamma$$

6. Angular rotations are positive clockwise when looking in the positive direction of the axis about which rotation occurs.

7. In *steady-state* condition, *lateral acceleration* is equal to the product of *centripetal acceleration* times the cosine of the vehicle's *sideslip angle.* Since in most test conditions the *sideslip angle* is small, for practical purposes, the *lateral acceleration* can be considered equal to *centripetal acceleration.*

8. Although the steering wheel is the primary directional control element it should be recognized that *longitudinal forces* at the wheels resulting from driver inputs to brakes or throttle can modify directional response.

9. Transient responses are described by the terminology normally employed for other dynamic systems. Some terminology is described in the "Control Engineers' Handbook,"[1] but a more complete terminology is contained in ANS C85.1-1963.[2]

10. Passenger vehicles exhibit varying characteristics depending upon test conditions and *trim.* Test conditions refer to vehicle conditions such as wheel loads, front wheel alignment, tire inflation pressure, and also atmospheric and road conditions which affect vehicle parameters. For example, temperature may change shock absorber damping characteristics and a slippery road surface may change tire cornering properties. *Trim* has been previously defined as the vehicle operating condition within a given environment, and may be specified in part by *steer angle, forward velocity,* and *lateral acceleration.* Since all these factors change the vehicle behavior, the vehicle *stability* must be examined separately for each environment and *trim.*

For a given set of vehicle parameters and particular test conditions, the vehicle may be examined for each theoretically attainable *trim.* The conditions which most affect *stability* are the *steady-state* values of *forward velocity* and *lateral acceleration.* In practice, it is possible for a vehicle to be stable under one set of operating conditions and unstable in another.

11. Divergent instability may be illustrated by operation above the *critical speed* of an *oversteer vehicle.* Any input to the steering wheel will place the vehicle in a turn of ever-decreasing radius unless the driver makes compensating motions of the wheel to maintain general equilibrium. This condition represents *divergent instability.* A linear mathematical analog of a vehicle is *divergently unstable* when its characteristic equation has any positive real roots.

12. Oscillatory instability may be illustrated by the *free control response* following a pulse input of displacement or force to the steering wheel. Some vehicles will turn first in one direction, and then the other, and so on, until the *amplitude* of the motion increases to the extent that the vehicle "spins out." In this event, the vehicle does not attempt to change its general direction of motion, but does not achieve a *steady-state* condition and has an oscillatory motion. A linear mathematical analog of a vehicle is *oscillatorily unstable* when its characteristic equation has any complex roots with positive real parts.

13. It is possible for a vehicle to be *understeer* for small inputs and *oversteer* for large inputs (or the opposite), as shown in Fig. 4, since it is a nonlinear system and does not have the same characteristics at all *trims.* Consequently, it is necessary to specify the range of inputs and velocities when making a determination of the vehicle's *steer characteristics.*

There is a set of equivalent definitions in terms of *yaw velocity* or curvature (reciprocal of radius of curvature), which can be used interchangeably with these definitions. These definitions only apply to two-axle vehicles, since the *Ackerman steer angle* only applies to two-axle vehicles.

[1] John G. Truxal (Ed.), Control Engineers' Handbook, New York: McGraw Hill.

[2] "Terminology for Automatic Control," ANS C85.1-1963, published by American Society of Mechanical Engineers.

14. *Ackerman Steer Angle Gradient* is equal to the wheelbase divided by the square of the vehicle speed (rad/ft/s^2).

15. For nonlinear steering systems, this ratio should be presented as a function of *steering wheel angle* in order to be compatible with the definition of *understeer/oversteer gradient.*

16. The *roll center* defined in 9.4.28 constitutes an idealized concept and does not necessarily represent a true instantaneous center of rotation of the sprung mass.

17. Illustration applies only for constant *overall steering ratio.* For other steering systems, refer directly to definitions for interpretation of data.

Appendix B
SAE J6a
Ride and Vibration Data Manual

Report of Riding Comfort Research Committee approved July 1946 and last revised by the Vehicle Dynamics Committee October 1965.

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Foreword

This is the third edition of Ride and Vibration Data to be issued by the SAE Vehicle Dynamics Committee, formerly the Riding Comfort Research Committee.

The first edition, prepared in 1945, was confined to basic relationships involved in vehicle suspension and impact energy absorption. However, the treatment of vibration did not go beyond the characteristics of undamped simple harmonic motion.

The second edition, issued in 1950, consisted essentially of the original material with the addition of a section on human vibration tolerance.

In this new edition, the editorial subcommittee has attempted to include a graphical presentation of damped vibrating system characteristics, aimed at applications to vehicle ride and vibration problems. A detailed description of the scope of this subject matter is given in the Introduction to Section 2.

SAE Vehicle Dynamics Committee:
William Le Fevre, Chairman

Editorial Subcommittee:
R. N. Janeway, Chairman, Janeway Engineering Co.
William Le Fevre, Le Fevre Co.
W. C. Oswald, Bostrom Research Labs.
John Versace, Ford Motor Co.

1. Basic Relationships

1.1 Acceleration versus Static Deflection

Fig. 1 shows the acceleration of a mass to which a force is applied through a spring. It expresses the relation A = x/δ, based on Newton's second law:

$$a = \frac{xk}{m}$$

$$A = \frac{a}{g} = \frac{xk}{mg} = \frac{x}{\delta}$$

where:

- A = Acceleration, g units
- a = Acceleration, in./sec^2
- g = Acceleration of gravity, 386 in./sec^2
- k = Spring rate, lb/in.
- m = w/g = mass, lb-sec^2/in.
- w = Weight of mass, lb
- δ = w/k = effective static deflection, in.
- x = Amplitude = deflection of spring from static equilibrium, in.

Referring to Fig. 1A, note that the dynamic deflection (amplitude) of the spring is the relative displacement of the ends of the spring and can be caused by the displacement of the mass or of the support or of both. Also note that the effective static deflection is not necessarily equal to the total loaded deflection. The relations expressed in this chart are not restricted to cyclic vibrations. They apply equally to the acceleration produced by a single sudden displacement. The figure refers to instantaneous values of acceleration and deflection, not to maxima only, for zero damping force.

The scales of Fig. 1B can be changed in conformity with the relation A = x/δ. Either x and δ can be multiplied by the same number, leaving A as given; or A and x can be multiplied by the same number, leaving δ as given.

EXAMPLE: A vehicle is sprung with 8 in. effective static deflection. The wheel is suddenly pushed up 2 in. What is the resulting acceleration?

ANSWER: At the intersection of the 8 in. vertical with the 2 in. amplitude curve read 0.25 g acceleration.

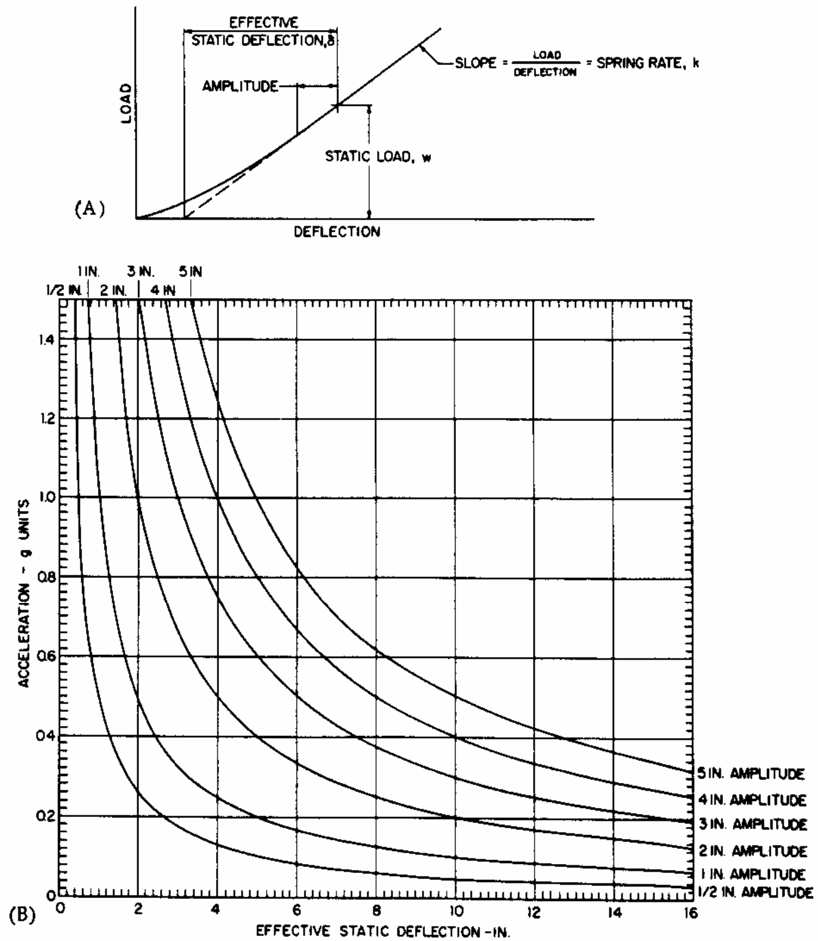


Fig. 1 - Acceleration versus static deflection at given amplitudes

1.2 Undamped Natural Frequency of Sprung Mass*

The fundamental physical quantity which determines the natural vibration frequency of an undamped single degree of freedom system is the acceleration of the suspended mass produced by unit displacement from its static position.

*See Refs. 1, 2, and 3.

This is developed to give the basic frequency equation in Fig. 2A. Note that the spring rate must be measured in the direction of the motion and at the static loaded position. The simplified system diagram in Fig. 2A assumes a constant spring rate. However, the effective static deflection is not necessarily equal to the total spring deflection at the static load, as illustrated in Fig. 1A.

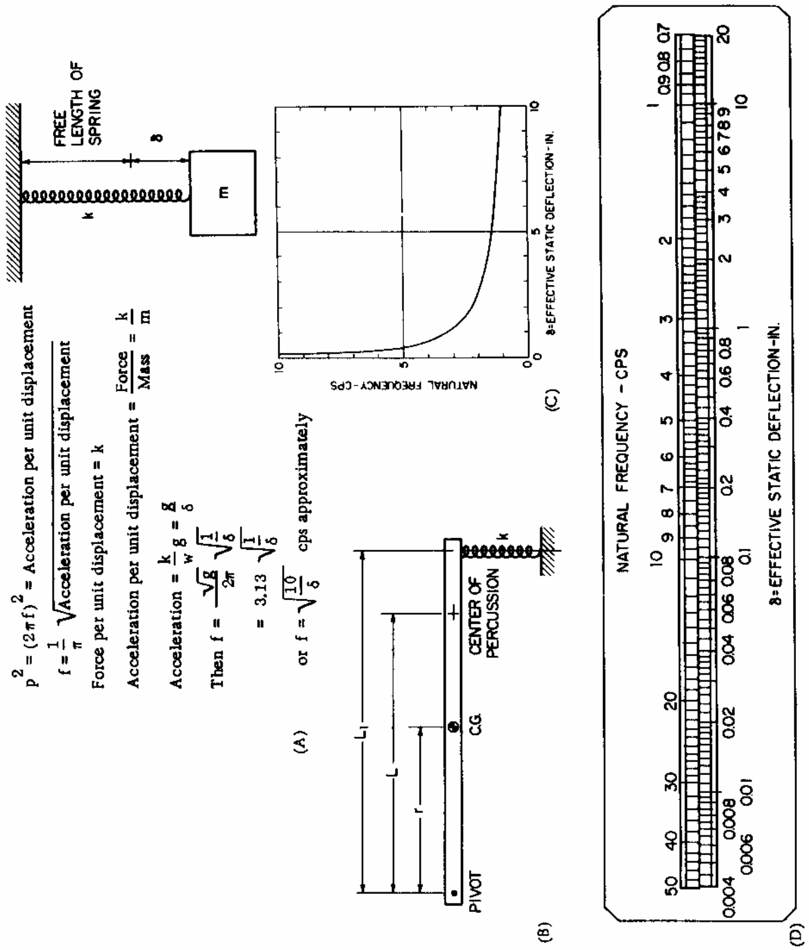


Fig. 2 - Undamped natural frequency versus static deflection

Fig. 2C shows the relation between static deflection and natural frequency in ordinary (Cartesian) coordinates; it give a visual impression of the characteristic.

Fig. 2D shows the same relationship by means of fixed logarithmic scales. It can be used to obtain quantitative values with satisfactory accuracy up to 50 cycles per second.

Fig. 2B illustrates another important type of system with single degree of freedom; namely, angular motion about a fixed pivot. In this case the acceleration per unit displacement is:

$$\ddot{\alpha} = \frac{\text{Restoring force moment per radian}}{\text{Mass moment of inertia}}$$

The deflection at the spring per radian $= L_1 =$ Distance from pivot to spring center.

The restoring force at the spring per radian $= kL_1$.

The restoring force moment per radian $= kL_1{}^2$.

The moment of inertia of the mass, m, about the pivot $= m\,(i^2 + r^2)$,

where:

i = Radius of gyration about the c.g.
r = Distance of c.g. from pivot

In this case it is useful to apply the concept of center of percussion which is located at the distance, L, from the pivot such that $L = r + i^2/r$.

Therefore, $r^2 + i^2 = rL$, and the moment of inertia = mrL.

Then, the frequency,

$$f = \frac{1}{2\pi}\sqrt{\ddot{\alpha}} = \frac{1}{2\pi}\sqrt{\frac{kL_1{}^2}{mrL}} = \frac{\sqrt{g}}{2\pi}\sqrt{\frac{kL_1{}^2}{wrL}}$$

Since the load on the spring, $R = \frac{wr}{L_1}$

the static deflection, $\delta = \frac{wr}{L_1 k}$

and, $f = \frac{\sqrt{g}}{2\pi}\sqrt{\frac{L_1}{\delta L}}$

Therefore, the frequency is again dependent on the static spring deflection but modified by the relation of the spring center to the center of percussion. It is important to note, however, that when $L_1 = L$, i.e., spring is located at the center of percussion, the frequency is determined by the static deflection alone, as in the case of rectilinear motion. Figs. 2C and 2D apply also to this special case of angular motion.

1.3 Relations in Simple Harmonic Motion

In simple harmonic motion, the distance, x, a vibrating mass is displaced from the static position at a given time, t, can be approximated closely by the equation:

$$x = x_0 \sin(2\pi f)t = x_0 \sin\omega t$$

where:

- x = Instantaneous displacement from static position, in.
- x_0 = Maximum displacement from static position (amplitude), in.
- t = Time from zero displacement, sec
- f = Frequency of oscillation, cps
- ω = $2\pi f$, radians per sec

From this equation, the following relationships for peak values can also be derived:

Maximum velocity: $v_m = 2\pi f x_0 = \omega x_0$ in./sec

Maximum acceleration: $a_m = 4\pi^2 f^2 x_0 = \omega^2 x_0$ in./sec^2

Maximum jerk (rate of change of acceleration): $j_m = 8\pi^3 f^3 x_0 = \omega^3 x_0$ in./sec^3

The interrelationships of frequency, velocity, amplitude, and acceleration are shown in Fig. 3.

Any deviation from true simple harmonic motion shows up more and more as higher derivatives of the motion are taken (or, as we go from displacement to velocity, to acceleration, to jerk).

EXAMPLE: A mass has an amplitude of 1 in. at a frequency of 1 cycle/sec. What is the maximum velocity and the maximum acceleration?

ANSWER: Follow the vertical line at frequency 1 to the intersection with the diagonal line for 1 in. amplitude. Reading horizontally to the left hand scale gives approximately 6.2 in./sec maximum velocity. Reading diagonally on the maximum acceleration scale gives approximately 0.1 g.

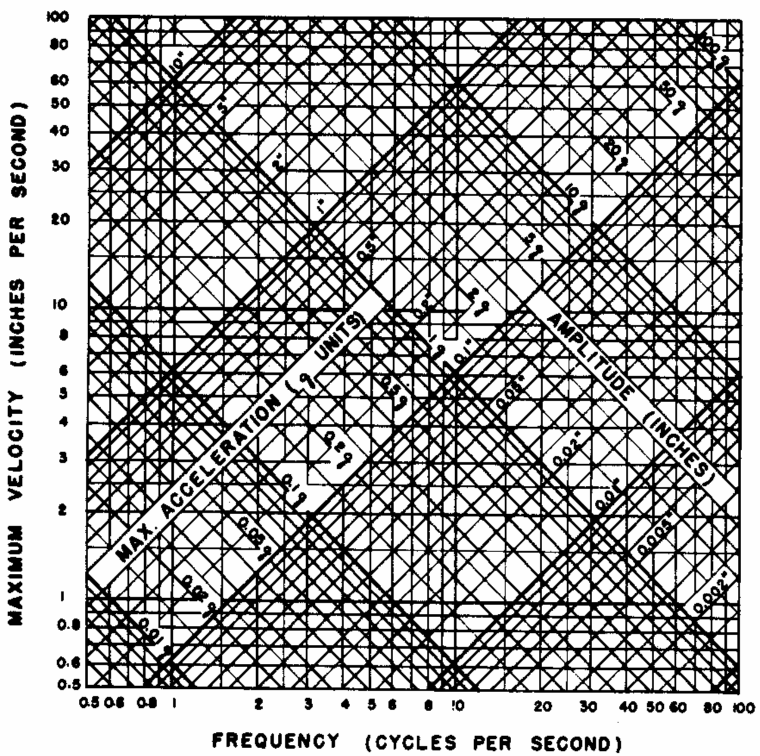


Fig. 3 - Relations in simple harmonic motion

1.4 Resonant Speed on Uniformly Spaced Road Disturbances*

Periodic force impulses, such as are produced by uniformly spaced road joints or washboard surfaces, can excite resonant vibration in a vehicle system if any natural frequency of the system is equal to, or an even multiple of, the impulse frequency. In the latter case, however, the induced vibration is usually of minor intensity because of the greater time available for damping between impulses.

It should be noted that the occurrence of this type of resonance is not related to the time duration of the individual impulse, but is determined by the uniform time period between impulses. Thus, impulse frequency, cpm = mph x 88/spacing (ft).

*See Ref. 5, p. 2-36.

For convenient reference, Fig. 4 has been prepared to relate spacing distance between disturbances and vehicle speed to impulse frequency. The lines are plotted with frequency as ordinate, from 50 to 700 cpm, and speed as abscissa, from 10 to 80 mph. Each line represents a periodic spacing, varying in steps from 3 to 60 ft. The ordinate scale is expanded below 100 cpm to enable more accurate reading of the chart in the passenger car range of sprung mass frequency.

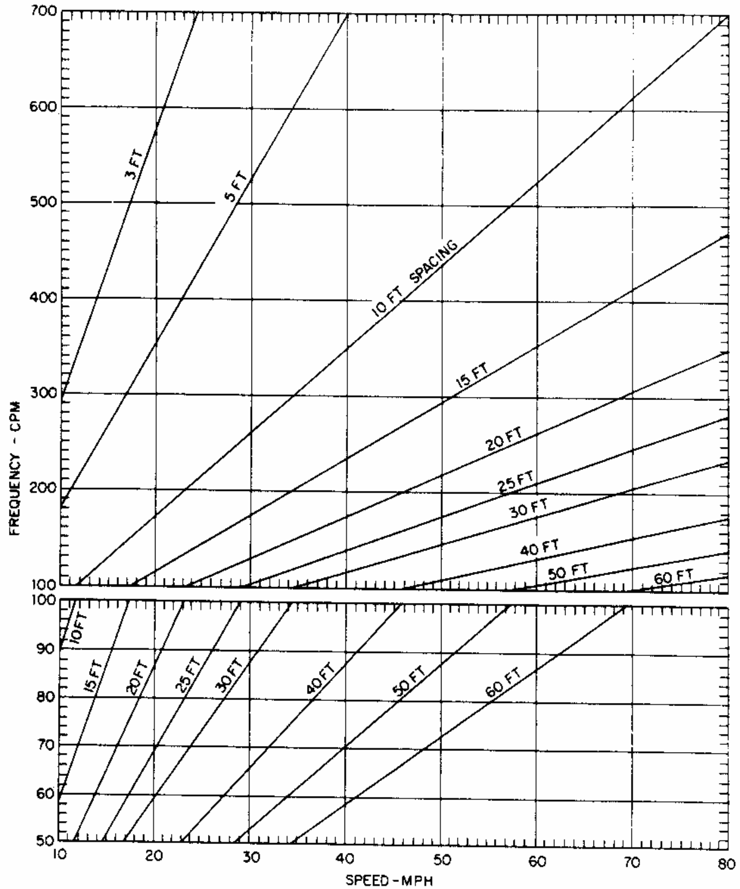


Fig. 4 - Resonant speed on uniformly spaced road disturbances

The following examples of the use of this chart have been selected to bring out some of the situations in which disturbing resonances are commonly experienced (all values are based on resonance with the fundamental of the road disturbance):

(a) A truck has a natural pitching frequency of 300 cpm of the entire mass on the tires alone, due to friction in the suspension. At what speed will resonance be excited by road joints uniformly spaced at 15 ft?

At intersection of the 15 ft line with the 300 cpm horizontal line, read 50 mph.

(b) A passenger car has a front end unsprung mass natural frequency of 11.7 cps or 700 cpm. At what speed will unsprung mass resonance occur on a road with 10 ft joint spacing?

At intersection of the 10 ft line with the horizontal line through 700 cpm, read 79.8 mph.

(c) The natural frequency in vertical bounce of a driver on the seat cushion of a truck is 240 cpm. At what speed is he likely to experience resonant bouncing on the cushion on a road with 20 ft joint spacing?

At intersection of the 240 cpm horizontal with the 20 ft line, read 55 mph.

It will be noted that none of these cases of resonance could occur in the operating speed range with uniform joint spacing of 25 ft or more. Unevenly spaced joints would largely eliminate the possibility of resonance.

2. Vibration Systems

This section presents the basic quantitative relationships in generalized vibration systems with a single degree of freedom. Both free and forced steady-state vibrations are described, including damping effects. Undamped conditions are automatically covered by the same equations, corresponding to zero value of the damping constant.

With the exception of the coulomb damping condition, only linear systems with viscous damping are considered. Such systems must meet the following conditions (refer to SAE J670, Vehicle Dynamics Terminology).

(a) The mass or masses have only one mode of vibratory motion, along a prescribed path. This is seldom strictly true, but the relations shown are sufficiently accurate if the constraining supports or guides are much stiffer than the elastic restraint along the path of motion.

(b) The elastic or spring restoring forces produced by displacement of the mass must be proportional to the displacement. Even though the spring rate may not be constant over the entire range of load, sufficient accuracy can usually be obtained by taking the average rate over the actual range of displacement.

(c) Since viscous damping is assumed, the damping force, by definition, is proportional to the relative velocity between the points of attachment of the damper.

In each case of forced vibration, a sinusoidal driving force is assumed. All the forces acting upon the system, then, can be represented graphically by rotating vectors. This type of diagram is used throughout to derive the basic relationships. The important advantages of this procedure are:

(a) Differential equations are entirely eliminated.
(b) The reader can readily visualize the mechanics of the vibration.

2.1 List of Symbols

2.1.1 Vibrating System Parameters

m = Mass, lb-sec^2/in.
k = Spring constant, lb/in.
F = Coulomb friction, lb
c = Viscous damping constant, lb-sec^2/in.
c_c = Critical damping constant, lb-sec^2/in.
= $2\sqrt{km}$
b = Damping ratio
= c/c_c

2.1.2 Vibratory Motion

t = Time, sec
f = Forcing frequency, cycles/sec
f_n = Undamped natural frequency, cps
f_d = Damped natural frequency, cps
ω = Vector angular velocity at forcing frequency = $2\pi f$, radians/sec

ω_n = Vector angular velocity at undamped natural frequency = $2\pi f_n$, radians/sec
ω_d = Vector angular velocity at damped natural frequency = $2\pi f_d$, radians/sec
a_0 = Amplitude of sinusoidal excitation of spring support, in.
X_0 = Initial mass amplitude in free vibration, in.
X_0' = Mass amplitude after one-half cycle in free vibration, in.
$X_1, X_2,..., X_n$ = Mass amplitude after successive cycles in free vibration, in.
ΔX = Mass amplitude decrement per cycle in coulomb damping, in.
x_0 = Mass amplitude in steady-state forced vibration, in.
x_{st} = Static deflection of mass corresponding to peak force steadily applied, in.
x'_{st} = Static deflection of mass corresponding to peak force at undamped natural frequency, in.
y_0 = Amplitude of relative motion between mass and vibrating spring support, in.
P_m = Peak value of sinusoidal exciting force, lb
P_0 = Peak value of sinusoidal exciting force, at undamped natural frequency, lb
P = Instantaneous variable sinusoidal force, lb
P_t = Peak value of force transmitted to the spring support, lb
ϕ = Phase angle

2.2 Free Vibration with Coulomb Damping*

Coulomb damping is produced when a constant friction force, F, opposes the vibratory motion. Referring to the force-displacement diagram (Fig. 5B), the decrement in amplitude can readily be derived from the energy relationships:

$$\Big[\text{The work done in accelerating the mass}\Big]_1^5 = k(X_0)^2/2 - FX_0$$

$$\Big[\text{The work done in decelerating the mass}\Big]_5^2 = k(X_0')^2/2 + FX_0'$$

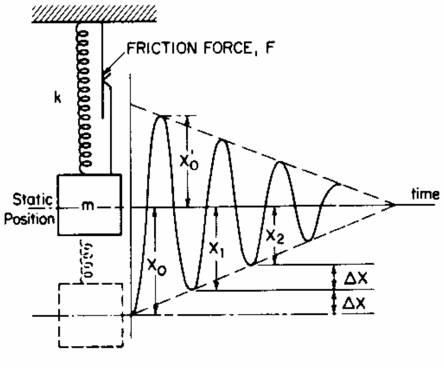
Since the vibrating mass starts from rest and again comes to rest after each half cycle:

$$k(X_0)^2/2 - FX_0 = kX_0'^2/2 + FX_0'$$

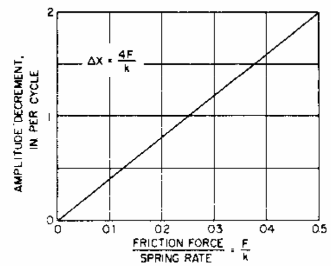
$$(k/2)[X_0^2 - X_0'^2] = F(X_0 + X_0')$$

$$X_0 - X_0' = 2F/k \text{ per } 1/2 \text{ cycle}$$

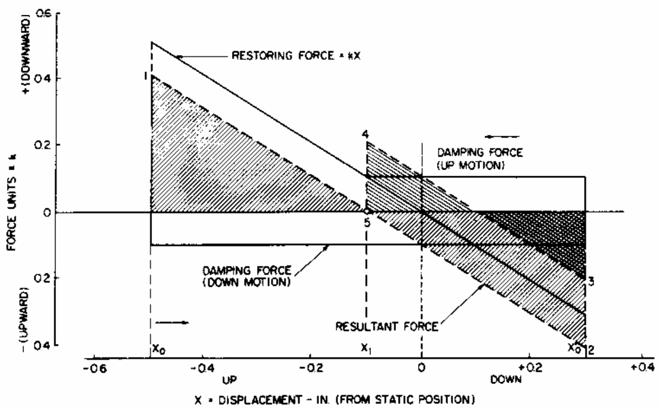
*See Ref. 4.



Conditions: initial deflection (upward) = 0.5 in.
Friction, lb = 0.10 x spring rate (k)



(A) Amplitude decrement



(B) Force - displacement diagram

Fig. 5 - Free vibration with coulomb damping

Thus, in free vibration, coulomb damping produces a constant decrement of amplitude per cycle, $\Delta X = 4F/k$, which depends only on the ratio of friction force to spring rate. Amplitude decrement in inches as a function of the ratio of damping friction force to spring rate is shown in Fig. 5A.

Referring again to the force-displacement diagram (Fig. 5B) the damping force is assumed to be 1/10 the spring force at unit-deflection or 1/10 the spring rate, k. The friction force opposes the spring force to produce the resultant force line 1-5-2 which, of course, determines the motion of the vibrating mass. Since the vibrating mass starts from rest, and comes to rest again after each half cycle, the positive work of acceleration must be equal to the negative work of deceleration represented by the shaded areas above and below the baseline. The same condition must be met on the return motion so that at the end of one cycle, the displacement is 1/10 in. (point 4) with a reduction in amplitude of 4/10 in. equal to four times the ratio of friction to spring rate in accordance with the above equation. The mass has, in one cycle, come to rest at a displacement of 1/10 in. because the spring restoring force and the friction force are equal at this position, and velocity is zero.

The force-displacement diagram also shows that the mass will come to rest at zero elongation of the spring, only if the initial displacement is an even multiple of 4F/k. The natural frequency of a free vibration determined by the equation in Fig. 3 is unchanged by friction damping.

2.3 Free Vibration with Viscous Damping*

Viscous damping is produced when the force opposing the motion is proportional to the velocity of the vibrating mass relative to the spring support. This type of damping in free vibration forms an amplitude decrement in which the amplitude between any two consecutive cycles is reduced by a constant ratio, according to the equation:

$$\frac{X_1}{X_0} = \frac{X_2}{X_1} = \frac{X_{(n+1)}}{X_n} = e^{-2\pi b/\sqrt{1-b^2}}$$

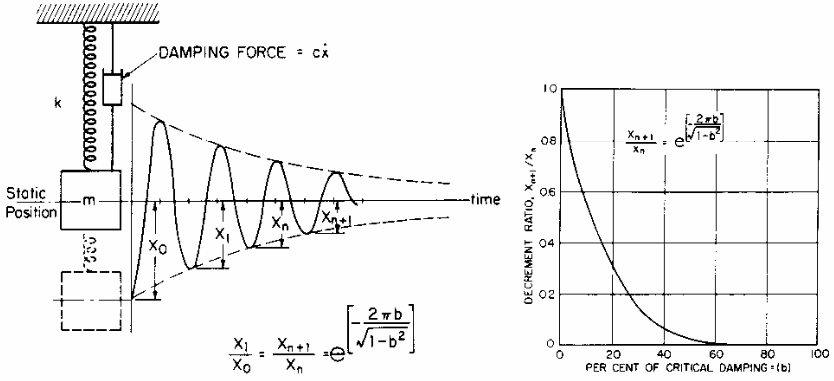
where:

$b = c/c_c$ = Damping factor which is the ratio of damping constant to the critical damping value, $c_c = 2\sqrt{km}$

$\frac{X_{(n+1)}}{X_n}$ = Amplitude ratio between any two consecutive cycles

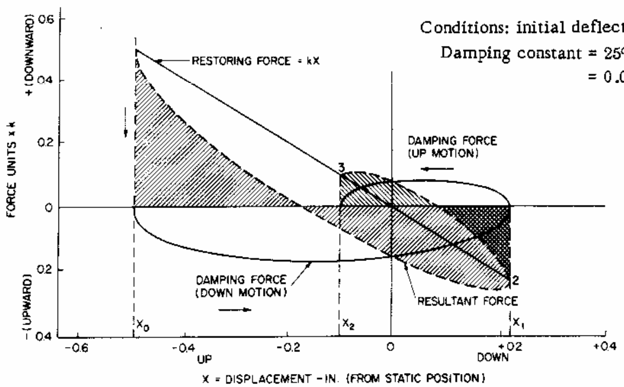
*See Ref. 4.

A typical displacement trace is illustrated in the graph in Fig. 6A. The amplitude decrement ratio as a function of percent of critical damping is shown plotted in the curve 6A.



(A) Amplitude decrement

Conditions: initial deflection (upward) = 0.5 in.
Damping constant = 25% critical
$= 0.025\,k$, lb sec/in. for $\frac{W}{k} = 1$



(B) Force-displacement diagram

Fig. 6 - Free vibration with viscous damping

Fig. 6B shows a force-displacement diagram for free vibration with viscous damping at 25% of critical. The damping force, in this case, is variable in contrast to a constant force for friction damping. The resultant force curve meets the condition that the work of acceleration equals the work of deceleration represented by the shaded areas in the diagram. Damping force is a

maximum at maximum velocity of the vibrating mass and is zero at points of maximum displacement where velocity is zero. Therefore, the maximum resultant deceleration force acting on the mass is out of phase with the maximum displacement.

After one complete cycle, the mass has momentarily come to rest at point 3, but oscillation will continue with a constant ratio in reduction of amplitude until displacement is small enough for extraneous friction to stop the motion.

Viscous damping reduces the natural frequency (f_d) below that of undamped free vibration (f_n) according to the equation:

$$f_d = \frac{1}{2\pi}\sqrt{\frac{k(1 - b^2)}{m}} \quad \text{or} \quad \frac{f_d}{f_n} = \sqrt{1 - b^2}$$

The effect on frequency is slight in the ordinary damping range. For example, if b = 0.25 (25% of critical damping) the frequency is reduced only 3%. However, when b = 1 (critical damping) the frequency becomes zero, indicating the absence of vibratory motion. Instead, the mass, when displaced, will gradually creep back to its neutral position without a change in direction of the spring force.

2.4 Forced Vibration with Viscous Damping*

2.4.1 Force Applied to Suspended Mass

2.4.1.1 Peak Exciting Force Constant — This case is represented by the system diagram (Fig. 7A) in which the suspended mass, m, is driven directly by a sinusoidal force, and is also acted upon by a viscous damper interposed between the mass and the spring support.

If the frequency of the driving force is f, and its maximum value is P_m, the generalized force can be written $P = P_m \sin(\omega t + \phi)$, where $\omega = 2\pi f$ and ϕ = phase angle by which maximum force leads the maximum displacement. The variable driving force can be graphically generated by a rotating vector of magnitude P_m, having a constant angular velocity = $2\pi f$, as in the Fig. 7B. Then, the vector component along the coordinate of vibratory motion at any instant equals the relative driving force. For steady-state vibration, the spring restoring force, inertia force, and damping force can also be represented by

*See Ref. 1, p. 48.

similar vectors having the proper relative magnitudes and phase relations. Note that the damping force, being proportional to, and in phase with, the mass velocity, is represented by a vector that lags the displacement by 90 deg.

For simplicity in deriving the equations, the vector diagram is drawn for the instant when the mass displacement is at its maximum positive value, ($\omega t = \pi/2$). Then for force equilibrium, the sum of vertical vector components and sum of horizontal vector components must each equal zero. Therefore,

Sum of vertical components

$$= kx_0 - m\omega^2 x_0 - P_m \cos\phi = 0 \tag{1}$$

Sum of horizontal components

$$= c\omega x_0 - P_m \sin\phi = 0 \tag{2}$$

In vertical force equation and Fig. 7B, note that inertia force is always in phase with displacement, while spring restoring force is opposed to displacement.

Combining Eqs. 1 and 2:

$$x_0 = \frac{P_m/k}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\frac{c}{c_c}\frac{\omega}{\omega_n}\right)^2}} \tag{3}$$

Since P_m/k is the deflection x_{st}, corresponding to the maximum force if steadily applied, the magnification factor is:

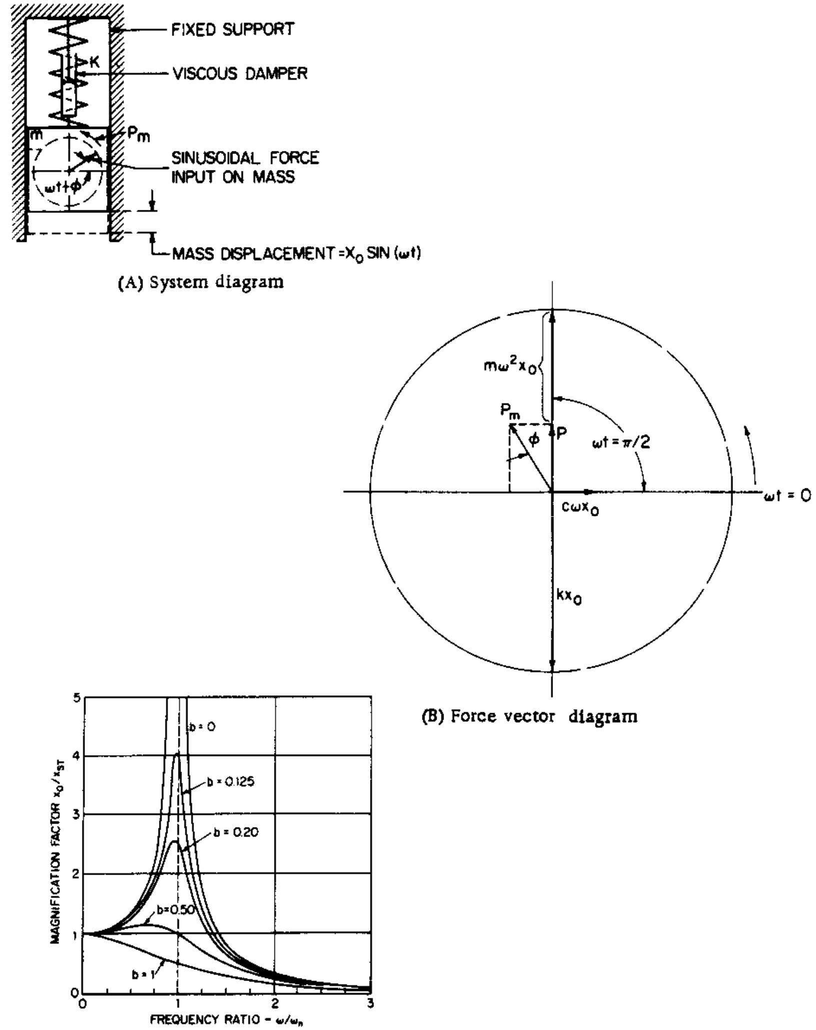
$$\frac{x_0}{x_{st}} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\frac{c}{c_c}\frac{\omega}{\omega_n}\right)^2}} \tag{4}$$

We may consider the undamped condition a special case in which c = 0. Then:

$$\frac{x_0}{x_{st}} = \frac{1}{1 - (\omega/\omega_n)^2} \tag{5}$$

Eq. 4 is the generalized magnification factor and is independent of the

magnitude of the applied force. Values of this factor are plotted in Fig. 7C against frequency ratio, for various constant damping intensities, expressed as the ratio, b, to the critical value.

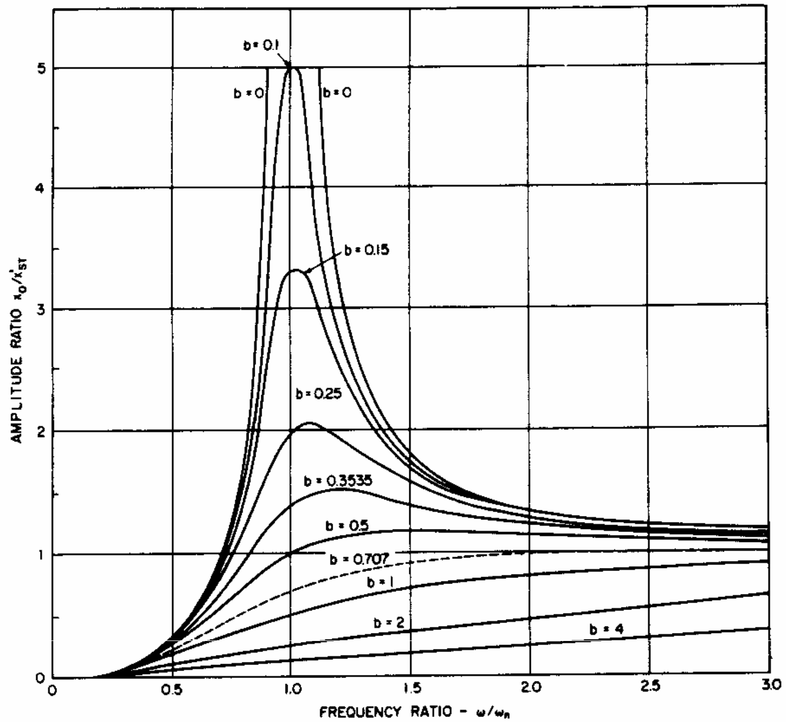


(A) System diagram

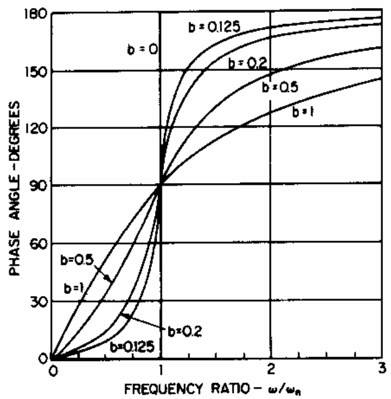
(B) Force vector diagram

(C) Magnification factor (constant peak driving force)

Fig. 7 - Forced vibration with viscous damping (driving force applied to suspended mass)



(D) Amplitude ratio when peak driving force is proportional to $(\omega/\omega_n)^2$



(E) Phase angle, ϕ

Fig. 7 - Continued

2.4.1.2 Exciting Force Increasing as Square of Frequency — Note that the magnification factor curves also define the amplitude ratio for the specific case where the maximum force remains constant as the frequency varies. However, another important type of forced vibration encountered in practice is that in which the maximum exciting force increases as the square of frequency. The most common situation of this kind occurs in machines having unbalanced rotating and/or reciprocating masses.

In this case, the maximum force may be written $P_m = P_0(\omega/\omega_n)^2$, where P_0 = maximum force at resonance, $(\omega/\omega_n = 1)$. Substituting for P_m in Eq. 3, we get for amplitude ratio:

$$\frac{x_0}{x'_{st}} = \frac{(\omega/\omega_n)^2}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2b\,\frac{\omega}{\omega_n}\right)^2}}$$

where:

$$x'_{st} = P_0/k.$$

Fig. 7D shows how the amplitude ratio varies with frequency for different damping intensities, in this case. It will be seen that, regardless of damping, the amplitude ratio is zero at zero frequency and approaches unity at infinite frequency.

Since ω_n is the undamped natural frequency by definition, resonance occurs at $\omega/\omega_n = 1$ only for very low damping values. It will be observed in Figs. 7C and 7D that the resonant frequency (corresponding to peak amplitude) departs progressively further from the undamped resonant frequency as damping intensity increases. Table 1 summarizes the equations which define the amplitude ratio at nominal resonance $(\omega/\omega_n = 1)$, the maximum amplitude, and the frequency of maximum amplitude as functions of damping ratio, for the two cases considered previously.

Referring again to Eqs. 1 and 2, the phase angle, ϕ, is readily derived by evaluating $\sin\phi$ and $\cos\phi$, and dividing $(\sin\phi)$ by $(\cos\phi)$. (x_0) drops out and we get:

$$\tan\phi = \frac{2b\,\omega/\omega_n}{1 - (\omega^2/\omega_n^2)}$$

Table 1

Force Condition	Amplitude Ratio at $\omega/\omega_h = 1$	Amplitude Ratio, Max	$\frac{\omega}{\omega_h}$ at Max Amplitude	Range of Damping Ratio
P_m = const.	$1/2b$	$\frac{1}{2b\sqrt{1-b^2}}$	$\sqrt{1-2b^2}$	$b < .707$
		1	0	$b \geq .707$
$P_m = P_0\left(\frac{\omega}{\omega_n}\right)^2$	$1/2b$	$\frac{1}{2b\sqrt{1-b^2}}$	$\frac{1}{\sqrt{1-2b^2}}$	$b < .707$
		1	∞	$b \geq .707$

Note that the phase angle is independent of the magnitude of the driving force, so that the curves of Fig. 7E apply equally to both cases considered. At $\omega/\omega_n = 1$, ϕ becomes 90 deg regardless of the degree of damping.

For values of ω/ω_n greater than 1, $\tan\phi$ becomes negative, corresponding to phase angles between 90 and 180 deg.

2.4.1.3 Equivalent Impedance — The impedance concept, as applied to mechanical vibrating systems, expresses the ratio of applied force to the induced motion at the point of force application. Thus, displacement impedance is the force required per unit of peak displacement; velocity impedance is the force required per unit of peak velocity.

The term "mobility" is used to designate the reciprocal of impedance. Thus, displacement mobility is the peak displacement per unit of driving force.

These quantities are obviously variable with frequency ratio, but are independent of the magnitude of the driving force.

For the case of vibration excited by a force applied directly to a suspended mass, the impedance and mobility functions become:

Displacement Impedance, $Z_d = P_m/x_0$

Velocity Impedance, $Z_v = P_m/x_0\omega$

Substituting the value of x_0 as a function of frequency ratio (Eq. 3):

$$x_0 = \frac{P_m}{k\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2b\frac{\omega}{\omega_n}\right)^2}}$$

$$Z_d = k\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2b\frac{\omega}{\omega_n}\right)^2}$$

$$Z_v = \frac{k}{\omega}\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2b\frac{\omega}{\omega_n}\right)^2}$$

Likewise, Displacement Mobility,

$$M_d = \frac{x_0}{P_m} = \frac{1}{k\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2b\frac{\omega}{\omega_n}\right)^2}}$$

Velocity Mobility,

$$M_v = \frac{\omega x_0}{P_m} = \frac{\omega}{k\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2b\frac{\omega}{\omega_n}\right)^2}}$$

Note that the displacement mobility varies with frequency in exactly the same way as the peak displacement for a constant harmonic force intensity.

2.4.2 Excitation Applied to Spring Support

2.4.2.1 Absolute Amplitude Ratio - When the system shown in Fig. 8A is excited by sinusoidal vibration of the spring support at a constant amplitude,

a_0, instead of by a force applied directly to the suspended mass, the absolute amplitude of the mass is obtained by deriving the equivalent force acting on the mass. It is important to note, in this case, that the spring force component due to the support motion acts on the mass in the same direction as the support displacement, as opposed to the spring force due to mass displacement.

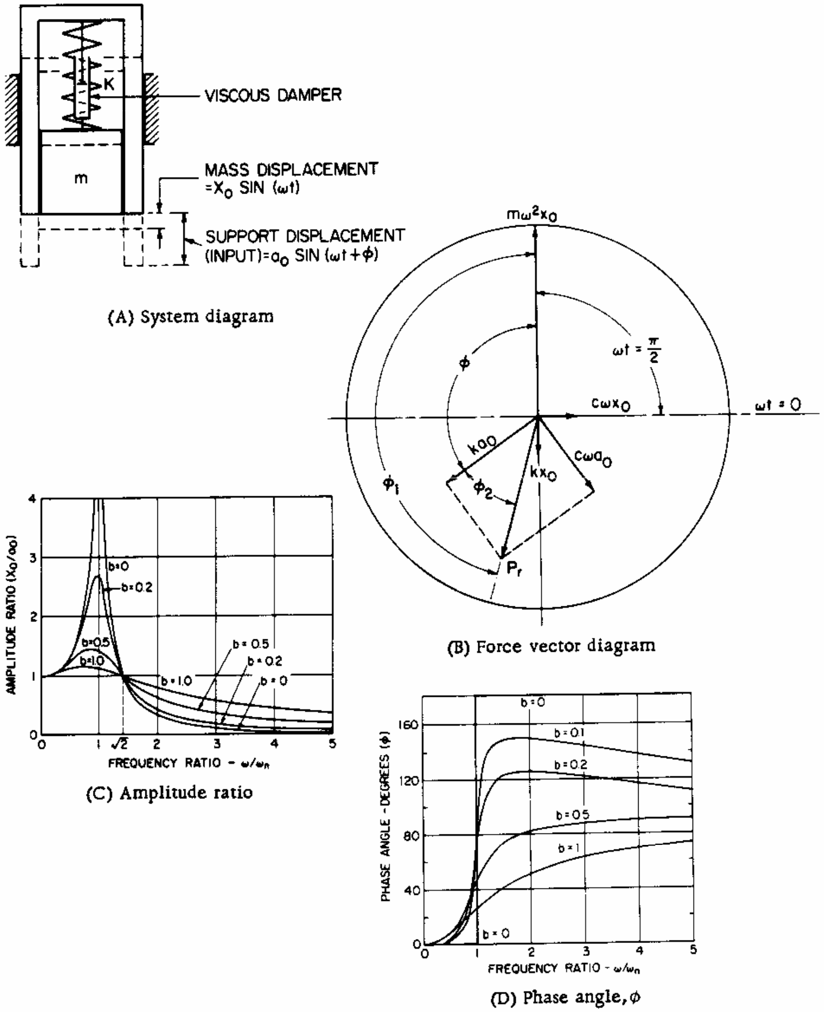


Fig. 8 - Forced vibration of spring support at constant amplitude (absolute amplitude ratio)

Referring to the vector diagram Fig. 8B (for the condition $\omega/\omega_n = 2$, $b = 0.2$), it will be seen that the rotating force vector, P_m, of Fig. 7B has been replaced by two rotating force vectors at 90 deg to each other, ka_0 and $c\omega a_0$, corresponding respectively to the spring force and to the damping force components of the exciting amplitude. The resultant of these forces is:

$$P_r = \sqrt{(ka_0)^2 + (c\omega a_0)^2} = a_0 k \sqrt{1 + \left(2b\frac{\omega}{\omega_n}\right)^2}$$

and the absolute amplitude of the mass, x_0, is determined by the same magnification factor as in Eq. 4, paragraph 2.4.1.1.

$$x_0 = P_r/k \qquad \text{(M.F.)}$$

Substituting in this expression the equation for P_r, the magnification factor yields:

$$\frac{x_0}{a_0} = \sqrt{\frac{1 + \left(2b\frac{\omega}{\omega_n}\right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2b\frac{\omega}{\omega_n}\right)^2}}$$

Referring to Fig. 8C, the values of this amplitude ratio are plotted for various damping intensities over a range of frequency ratios. It is significant to note that damping is beneficial in reducing amplitude only for frequency ratios lower than $\sqrt{2}$. At higher ratios the amplitude increases progressively with the damping intensity. However, with zero damping the amplitude ratio reduces to the same equation as that for constant force excitation applied directly to the mass (Eq. 4). Thus, the curves for b = 0 are identical in Figs. 7C and 8C.

The general expression for the phase angle between the mass and support motions is derived as follows (refer to Fig. 8B):

ϕ_1 = Phase angle between P_r and x_0

ϕ_2 = Phase angle between P_r and a_0

ϕ = Phase angle between a_0 and $x_0 = \phi_1 - \phi_2$

$$\tan \phi_1 = \frac{2b\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$ (See paragraph 2.4.1.2)

$$\tan \phi_2 = \frac{ca_0\omega}{a_0 k} = \frac{c\omega}{k} = 2b\left(\frac{\omega}{\omega_n}\right)$$ (See Fig. 8B)

$$\tan \phi = \tan(\phi_1 - \phi_2) = \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2}$$

Substituting,

$$\tan \phi = \frac{2b\left(\frac{\omega}{\omega_n}\right)^3}{1-\left(\frac{\omega}{\omega_n}\right)^2 + \left(2b\frac{\omega}{\omega_n}\right)^2}$$

This equation is shown plotted in Fig. 8D for various frequency ratios and for different damping intensities.

The frequency ratio at which maximum amplitude occurs is given by the expression:

$$\left(\frac{\omega}{\omega_n}\right)_m = \frac{1}{2b}\sqrt{\sqrt{1+8b^2} - 1}$$

For finite values of b, $(\omega/\omega_n)_m$ is less than 1, but approaches unity as b approaches zero.

The maximum amplitude ratio for a given damping intensity, b, may be determined directly from the relation:

$$\left(\frac{x_0}{a_0}\right)_m = \sqrt{\frac{\sqrt{1+8b^2}}{\left\{1 - \frac{1}{4b^2}\left[\sqrt{1+8b^2} - 1\right]\right\}^2 + \sqrt{1+8b^2} - 1}}$$

For small values of b (less than 0.15) this equation can be reduced to the approximate form:

$$\left(\frac{x_0}{a_0}\right)_m = \sqrt{\frac{\sqrt{1+8b^2}}{\sqrt{1+8b^2} - 1}}$$

2.4.2.2 Relative Amplitude Ratio — Fig. 9 illustrates the method of deriving the equations for the relative motion between vibrating mass and support in the same system as in Fig. 8, where the excitation is applied to the support. Knowledge of the relative motion is needed in such important practical problems as designing vibration instruments and providing for clearance between components of a vibrating system.

Referring to the vector diagram in Fig. 8B, it can be seen that the separate spring and damping force vectors associated with the respective motions of mass and support can be combined into resultants as shown in Fig. 9B, in terms of the relative motion, y, between the mass and support. The resultant spring and damping forces are necessarily in 90 deg phase relationship, while the inertia force on the mass remains in phase with the absolute mass displacement. In Fig. 9C, the complete force vector diagram is seen to reduce to the simple form in which the only forces acting on the mass are the inertia force due to its absolute motion, the spring restoring force due to the relative displacement between mass and support, and the damping force due to the relative velocity between the mass and support. It is clear that for equilibrium the resultant between the spring and damping forces must be equal and opposite to the inertia force. The equations are readily derived as follows:

Denoting the phase angle between the relative motion and absolute motion as ϕ_3 and summing up the vertical and horizontal forces when the relative displacement is a maximum (y_0), (when $\omega t = \pi/2$):

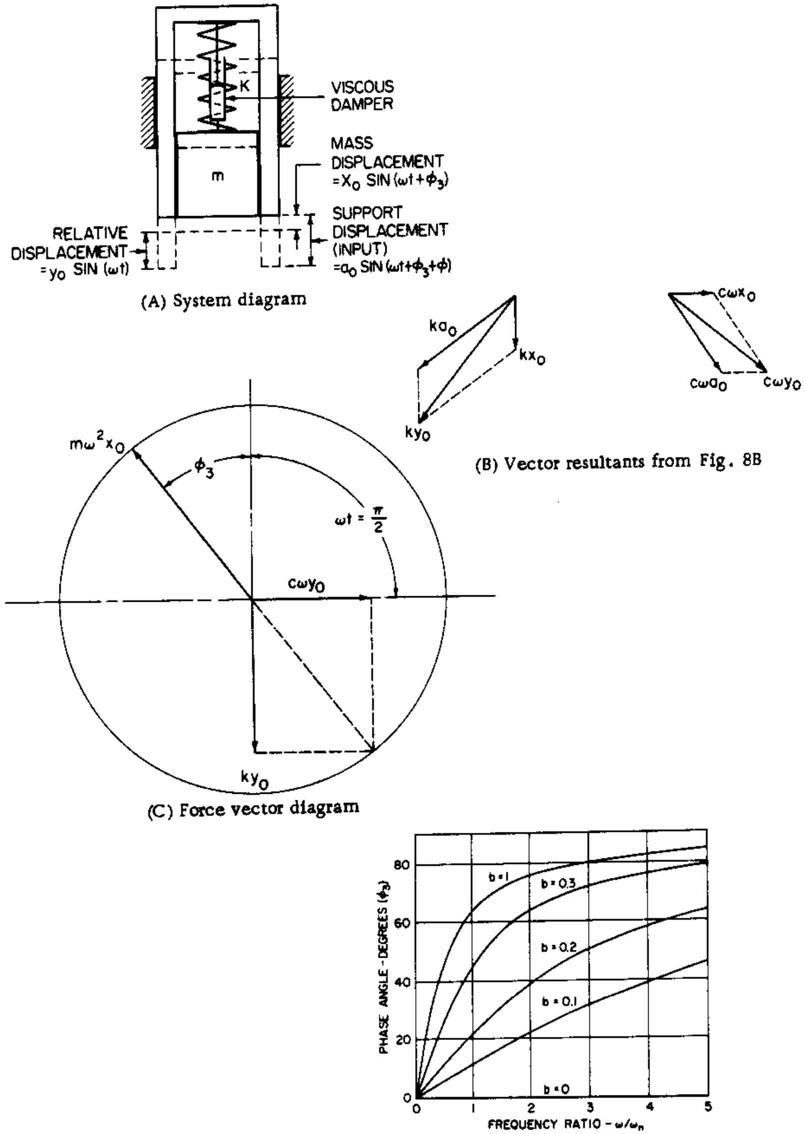


Fig. 9 - Forced vibration of spring support at constant amplitude (relative motion between mass and support)

Sum of the vertical forces

$$-m\omega^2 x_0 \cos\phi_3 + ky_0 = 0$$

therefore:

$$\cos\phi_3 = \frac{ky_0}{m\omega^2 x_0}$$

Sum of the horizontal forces

$$-m\omega^2 x_0 \sin\phi_3 + c\omega y_0 = 0$$

therefore:

$$\sin\phi_3 = \left[\frac{c\omega y_0}{m\omega^2 x_0}\right]$$

Since $\sin^2\phi_3 + \cos^2\phi_3 = 1$:

$$\left[\frac{c\omega y_0}{m\omega^2 x_0}\right]^2 + \left[\frac{ky_0}{m\omega^2 x_0}\right]^2 = 1$$

$$y_0^2 = \frac{(x_0 m\omega^2)^2}{k^2\,[1 + (c\omega/k)^2]}$$

$$y_0 = \frac{x_0\omega^2/\omega_n^2}{\sqrt{1 + (c\omega/k)^2}}$$

From paragraph 2.4.2.1, the absolute amplitude ratio is:

$$\frac{x_0}{a_0} = \sqrt{\frac{1 + \left(2b\dfrac{\omega}{\omega_n}\right)^2}{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(2b\dfrac{\omega}{\omega_n}\right)^2}}$$

so:

$$\frac{y_0}{a_0} = \frac{\omega^2/\omega_n^{\,2}}{\sqrt{(1 - \omega^2/\omega_n^{\,2})^2 + (2b\omega/\omega_n)^2}}$$

Note that the expression for relative amplitude ratio is identical with that in paragraph 2.4.1.2, when the mass is excited directly by a force varying as the square of the frequency ratio. Thus, the curves of Fig. 7D apply to this case.

$$\tan\phi_3 = \sin\phi_3 / \cos\phi_3 = c\omega/k = 2b\,\omega/\omega_n$$

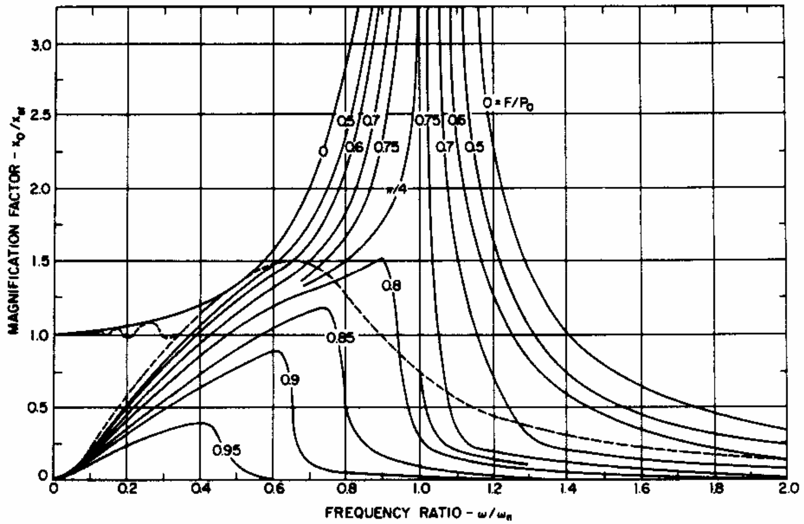
This phase angle relationship is shown in Fig. 9D for various damping intensities and frequency ratios.

2.5 Forced Vibration with Coulomb Damping

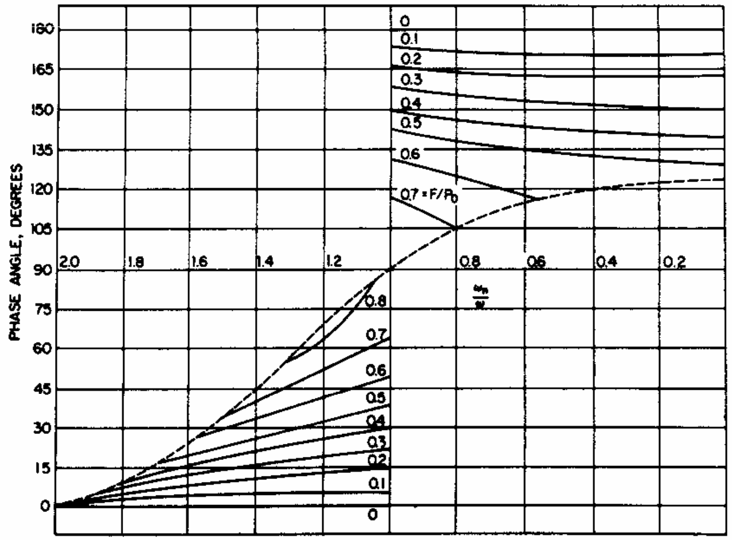
As pointed out earlier in this section, a system with coulomb (constant friction) damping is nonlinear and the solution of the differential equations becomes complicated. Den Hartog (Ref. 1) has arrived at an exact solution, which gives amplification factors and phase angles as shown in Fig. 10A and B, for specified ratios of friction to maximum driving force (F/P_0), over a range of frequency ratios (ω/ω_n).

Den Hartog points out that the dotted line in Fig. 10A defines the limits below which the vibratory motion is not continuous, but involves a "stop" or dwell every half-cycle. His analysis also emphasizes an important limitation of coulomb damping in that it will not restrict the amplitude of resonant vibration unless the friction force is greater than $\pi P_0/4$. This can be seen from the fact that the work done by a harmonic driving force at resonance is equal to $\pi P_0 x/4$, whereas the work done by the constant friction force, F, is equal to Fx. Consequently, if F is less than $\pi P_0/4$, the amplitude will increase on successive cycles to infinity. Coulomb damping is not applicable to a system which is subjected to steady-state vibration at resonance unless the maximum value of the driving force has a known limitation.

An approximate solution for this case is also given in the cited reference, but its use is not recommended because of the very limited range of conditions for which it is valid.



(A) Magnification factor



(B) Phase angle

Fig. 10 - Forced vibration with coulomb damping

An interesting approximate solution for the use of combined viscous and friction damping is given by Timoshenko (Ref. 3, p. 96). This solution is applicable when the friction damping is small relative to the viscous component.

2.6 Force Transmission Through Suspension

2.6.1 Direct Excitation of Mass with Viscous Damping

2.6.1.1 Constant Peak Driving Force — From the standpoint of vibration isolation, when a directly excited mass is flexibly mounted, we are interested in the magnitude of the force transmission to the support. Thus, referring to the force vector diagram Fig. 7B for the forced vibration system with constant maximum driving force and viscous damping, it will be seen that the spring force, kx_0, and damping force $c\omega x_0$, both react on the support. Since the two force components are 90 deg out of phase, their vector resultant as shown in Fig. 11A is

$$P_t = \sqrt{(x_0 k)^2 + (c\omega x_0)^2} = x_0 k\sqrt{1 + (c\omega/k)^2}$$

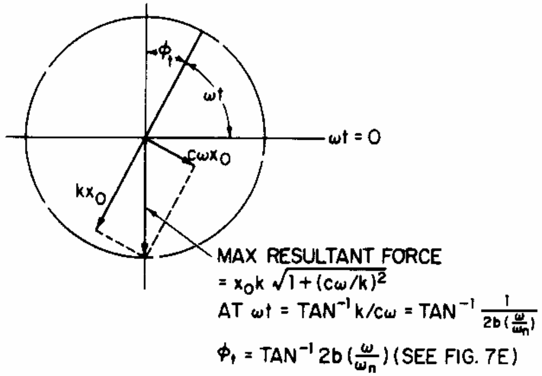
$$= x_0 k\sqrt{1 + (2b\omega/\omega_n)^2}$$

Since (from Eq. 3, paragraph 2.4.1.1):

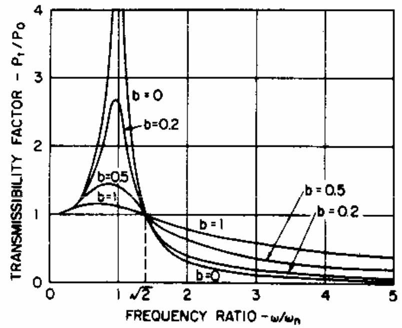
$$x_0 = \frac{P_m/k}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2b\frac{\omega}{\omega_n}\right)^2}}$$

$$P_t = \frac{P_m\sqrt{1 + (2b\omega/\omega_n)^2}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2b\frac{\omega}{\omega_n}\right)^2}}$$

This resultant is obviously the maximum force transmitted to the support when $\omega t = \tan^{-1} k/c\omega = \tan^{-1} 1/2b\,(\omega/\omega_n)$. This is the position shown in Fig. 11.



(A) Condition for maximum transmitted force



(B) Transmissibility factor

Fig. 11 - Force transmission through suspension
CASE 1: Constant maximum driving force on mass, with viscous damping

Thus, the transmissibility, defined as the ratio of maximum transmitted force to driving force is,

$$\frac{P_t}{P_m} = \sqrt{\frac{1 + (2b\omega/\omega_n)^2}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2b\frac{\omega}{\omega_n}\right)^2}}$$

This equation is shown plotted in Fig. 11B and will be seen to be identical with that for absolute amplitude ratio, when a constant vibration amplitude is applied to the support, Fig. 8C.

It will be noted that damping reduces the transmissibility for frequency ratios less than $\sqrt{2}$ but increases it for higher ratios. Thus, the undamped system has smaller transmissibility for frequency ratios larger than $\sqrt{2}$.

The force vector diagram, Fig. 11A, shows that the transmitted force reaches its maximum value at $\omega t = \tan^{-1} 1/2b(\omega/\omega_n)$, and at 180 deg intervals thereafter. Since the mass displacement is a maximum at $\omega t = \pi/2$ or $3\pi/2$, the phase angle, ϕ_t, by which the transmitted force leads the mass displacement is:

$$\phi_t = \frac{\pi}{2} - \tan^{-1} \frac{1}{2b\left(\dfrac{\omega}{\omega_n}\right)} = \tan^{-1} 2b\left(\frac{\omega}{\omega_n}\right)$$

This is identical with the phase relationship previously derived between relative displacement and mass displacement under constant amplitude excitation of the support. Therefore, the curves of Fig. 9D also define the value of ϕ_t, as a function of frequency ratio and damping intensity.

2.6.1.2 Driving Force on Mass Proportional to Square of Speed with Viscous Damping* — Instead of a constant driving force, let the driving force on the mass increase with the square of the frequency ratio so that $P = P_0 (\omega/\omega_n)^2$, where P_0 is the maximum driving force at resonance ($\omega/\omega_n = 1$).

Although the transmissibility (ratio of transmitted force to applied force) remains the same as in Case 1 (Fig. 11), the absolute value of transmitted force obviously varies in the same ratio as the applied force, or as $(\omega/\omega_n)^2$. The transmitted force thus becomes

$$P_t = P_0 (\omega/\omega_n)^2 \times \text{transmissibility}$$

$$\frac{P_t}{P_0} = \left(\frac{\omega}{\omega_n}\right)^2 \sqrt{\frac{1 + (2b\omega/\omega_n)^2}{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(2b\dfrac{\omega}{\omega_n}\right)^2}}$$

*See Ref. 2.

Fig. 12 shows the ratio of force transmission, P_t/P_0, plotted against frequency ratio, ω/ω_n, for various damping intensities. Comparison with Fig. 7D shows that the amplitude gives no indication of the detrimental effect of even a small amount of viscous damping at frequency ratios greater than $\sqrt{2}$, when the applied force is a function of the square of frequency. It is evident that at high values of ω/ω_n, the force transmitted to the support is primarily through the damper.

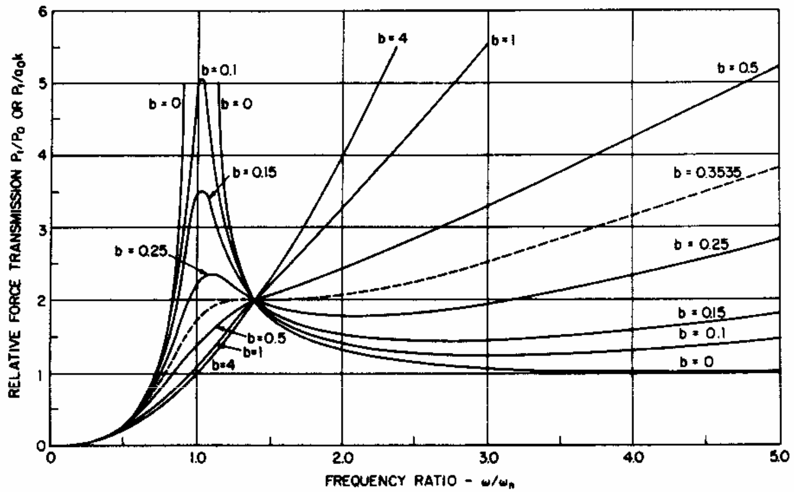


Fig. 12 - Transmission through suspension
CASE 2: Driving force on mass proportional to square of speed (T.F. = P_t/P_0) or spring support excited at constant amplitude (T.F. = P_t/a_0k)

In the case of vibration excited by oscillation of the spring support at constant amplitude, the transmitted force vector is likewise equal to the resultant of the relative spring and damping force vectors. In Fig. 9C this resultant, P_t, is shown to be equal to the inertia force, $m\omega^2 x_0$. By substituting for x_0, its value as derived in paragraph 2.4.2.1, we get:

$$P_t = m\omega^2 a_0 \sqrt{\frac{1 + (2b\omega/\omega_n)^2}{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(2b\dfrac{\omega}{\omega_n}\right)^2}}$$

The maximum force transmission, relative to (a_0k) as nominal input force, becomes:

$$\frac{P_t}{a_0k} = \frac{\omega^2}{\omega_n^2}\sqrt{\frac{1+(2b\omega/\omega_n)^2}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2b\frac{\omega}{\omega_n}\right)^2}}$$

It is evident that the force reaction formula so derived is identical with the formula given above for directly applied excitation to the suspended mass variable as the square of the frequency. Therefore, the curves of Fig. 12 apply equally when the spring support is excited at constant amplitude. The corresponding maximum acceleration of the sprung mass is:

$$A_m = \frac{P_t}{m} = a_0\omega^2\sqrt{\frac{1+(2b\omega/\omega_n)^2}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2b\frac{\omega}{\omega_n}\right)^2}}$$

This relationship is fundamental to the ride problem, since it covers the condition of a vehicle traversing a series of periodic sine wave irregularities of constant amplitude at variable speed.

2.6.1.3 Equivalent Impedance — When excitation is applied to the spring support, the input force must be an equal reaction to the force transmitted to the mass. The impedance relationships, thus, become:

(Displacement)

$$Z_d = \frac{P_t}{a_0} = m\omega^2\sqrt{\frac{1+(2b\omega/\omega_n)^2}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2b\frac{\omega}{\omega_n}\right)^2}}$$

(Velocity)

$$Z_v = \frac{P_t}{a_0\omega} = m\omega\sqrt{\frac{1+(2b\omega/\omega_n)^2}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2b\frac{\omega}{\omega_n}\right)^2}}$$

The corresponding mobility equations, as reciprocals, are:

$$M_d = \frac{1}{m\omega^2}\sqrt{\frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2b\frac{\omega}{\omega_n}\right)^2}{1+(2b\omega/\omega_n)^2}}$$

$$M_v = \frac{1}{m\omega}\sqrt{\frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2b\frac{\omega}{\omega_n}\right)^2}{1+(2b\omega/\omega_n)^2}}$$

It is important to note that the above equations are entirely different from the corresponding expressions in paragraph 2.4.1.3 for the case of driving force applied directly to the mass. This comparison brings out the wide variability of the impedance characteristic of a vibrating system, depending on where the driving force is applied.

*2.6.2 Comparison of Viscous and Coulomb Damping**

To compare the effects of viscous and coulomb damping, assume the following alternative conditions (driving force on mass assumed proportional to square of frequency, $P_m = P_0(\omega/\omega_n)^2$ in both cases):

(a) The friction force of coulomb damping is equal to the driving force at resonance ($\omega/\omega_n = 1$), $F = P_0$

*Ref. 4

(b) Viscous damping is 25% of critical

These conditions are chosen to give about the same transmissibility in the resonant frequency range.

In Fig. 13A, the undamped and viscous damping curves have been transposed from Fig. 7D. The coulomb damping curve is obtained by interpolation from the curves of Chart 10A, as follows:

Up to $\omega/\omega_n = 1$, for $F/P_0 = 1$, amplitude = 0.

Taking $F = P_0$:

$$\text{For } \frac{\omega}{\omega_n} > 1,\ P_m = P_0\left(\frac{\omega}{\omega_n}\right)^2 \text{ and } \frac{F}{P_m} = \frac{P_0}{P_0\left(\frac{\omega}{\omega_n}\right)^2} = \left(\frac{\omega_n}{\omega}\right)^2$$

From Fig. 10A, value of x_0/x_{st} is obtained for a given value of ω/ω_n at intersection with curve corresponding to $F/P_0 = F/P_m = (\omega_n/\omega)^2$.

$$\text{Since } x'_{st} = P_0/k,\ x_{st} = P_m/k = (\omega/\omega_n)^2\, P_0/k = (\omega/\omega_n)^2\, x'_{st}$$

$$\text{Then } x_0/x'_{st} = (x_0/x_{st})\ (\omega/\omega_n)^2$$

At frequency ratios above 2.5, there is no difference in displacement amplitude for either viscous or coulomb damping or for no damping. This comparison gives no indication of the force transmission to the support through the suspension, especially for viscous damping.

As with viscous damping, the same curves apply to vibration excited by sinusoidal motion of the support at constant amplitude a_0. In this case, however, the ordinate scale gives amplitude ratio in terms of (y_0/a_0), where y_0 is the relative displacement amplitude between mass and support (see paragraph 2.4.2.2).

Fig. 13B shows the comparative force transmission. For direct excitation of the mass, the ordinate scale defines the force transmitted to the support in terms of the ratio P_t/P_c; for excitation of the support the ordinate scale gives the ratio P_t/a_0k, as in Fig. 12. It can be seen that the undamped suspension, although capable of infinite amplitude at resonance, gives the lowest force transmission at high frequency ratios.

With viscous damping at 25% of critical, the force transmission at high frequencies increases continuously above a frequency ratio of about 2.0.

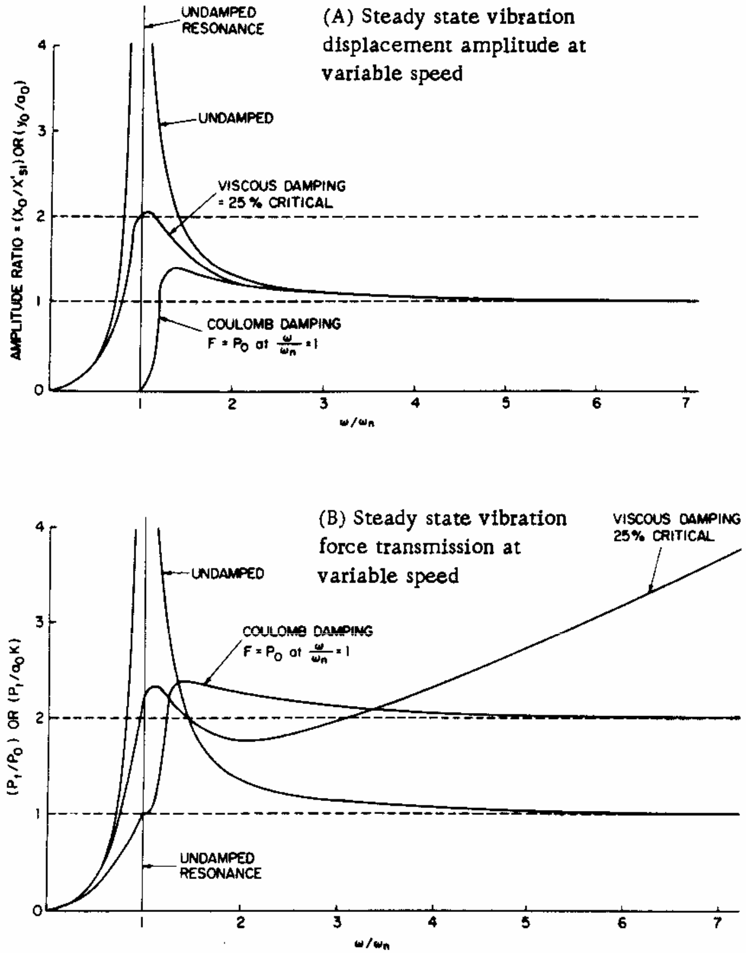


Fig. 13 - Comparison of viscous and coulomb damping on displacement amplitude and force transmission

With the specified coulomb damping, the maximum force transmission at resonance is nearly the same as for viscous damping but thereafter reduces continuously with increase in frequency ratio, approaching $2P_0$ as a limit. Thus at high frequency ratios, coulomb damping adds to the force transmission only by the amount of the damping friction force.

In general, it can be said that the optimum damping characteristic for any given suspension is a compromise between a minimum penalty of force transmission and the required amplitude control.

2.6.3 Vibrating System with Two Degrees of Freedom

The previous section is confined to consideration of systems having a single degree of freedom. A great many practical vibration problems, however, involve two degrees of freedom. In keeping with the definition in the introduction to Section 2, this means that the system either contains two masses, each capable of a single type of independent motion, or a single mass capable of two independent types of motion.

A primary example of the latter configuration, which is basic to ride problems, is a vehicle mass having two spaced spring supports. Considering only degrees of freedom in the longitudinal plane, the mass has two independent modes of motion, namely, vertical displacement of the center of gravity and angular displacement about the center of gravity.

In order to present in a useful form the effects of the principal parameters, the standard equations for frequencies and centers of oscillation (Ref. 3, p. 201) have been reduced to a few dimensionless terms.

Thus, if r = ratio of static spring deflection at the supports = δ_2/δ_1 where $\delta_2 \geq \delta_1$

$d = i^2/AB$ = dynamic index

where:

- i = Radius of gyration about c.g.
- A = Distance of c.g. from spring support at δ_1
- B = Distance of c.g. from spring support at δ_2
- $a = A/(A+B)$

Then, if

- f_1 = Bounce frequency (about center outside of spring supports), cycles per second
- f_2 = Pitch frequency (about center between spring supports), cycles per second
- m/B = Relative distance from c.g. to pitch center ((-) toward δ_2)
- n/A = Relative distance from c.g. to bounce center ((+) toward δ_1)

NOTE: For values of d > 1 bounce and pitch frequencies and centers are reversed.

Then

$$f_1^2 = 4.9/\delta_2 \quad [C - \sqrt{D^2 + E}]$$

$$f_2^2 = 4.9/\delta_2 \quad [C + \sqrt{D^2 + E}]$$

where:

$$C = a\left(\frac{1}{d} - 1\right)(r-1) + \frac{1}{d} + r$$

$$D = a\left(\frac{1}{d} + 1\right)(r-1) + \frac{1}{d} - r$$

$$E = \frac{4}{d}\, a\,(1-a)\,(r-1)^2$$

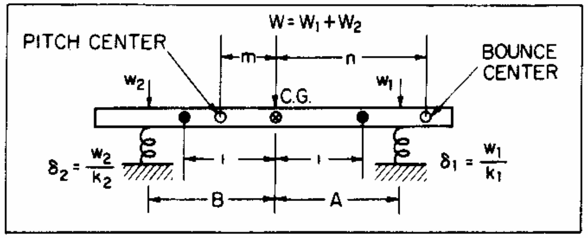
Likewise,

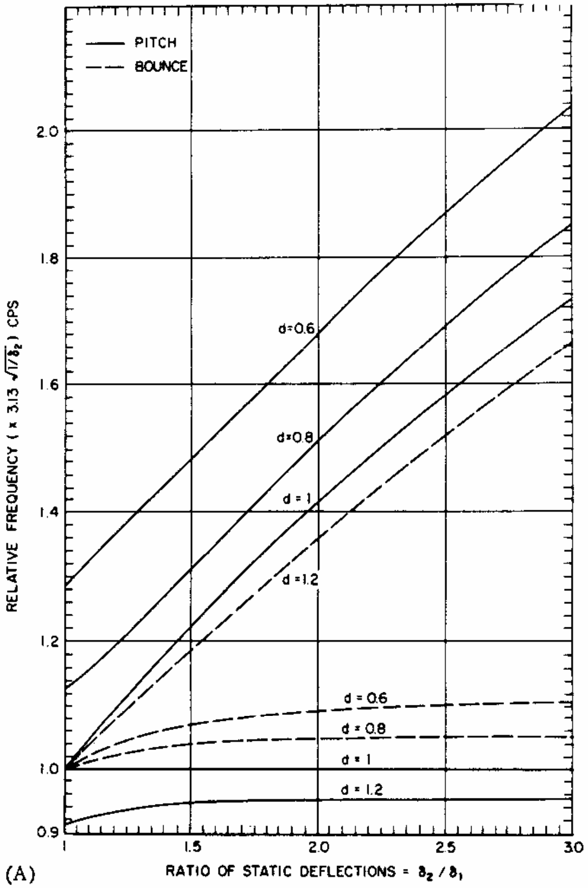
$$\frac{m}{B} = \frac{2a\,(1-r)}{D + \sqrt{D^2 + E}} \qquad \frac{n}{A} = \frac{2\,(1-r)\,(1-a)}{D - \sqrt{D^2 + E}}$$

Note that $-(m/B \bullet n/A) = d = i^2/AB$, a useful relationship for checking purposes.

The curves of Fig. 14 show in relative terms the characteristic changes in frequencies and centers of oscillation with variation in the ratio δ_2/δ_1, for four values of dynamic index, namely, d = 1.2, 1, 0.8, and 0.6. The chart values have been calculated for a center of gravity position equidistant from the spring centers (a = 0.5). Nevertheless, these curves are applicable to the usual passenger car range of c.g. position (a = 0.5 to 0.6), with negligible error as to frequencies, and without exceeding ±5% error in centers of oscillation. However, for accurate determination in commercial vehicles, where the c.g. usually falls outside this range, the values should be calculated from the general equations.

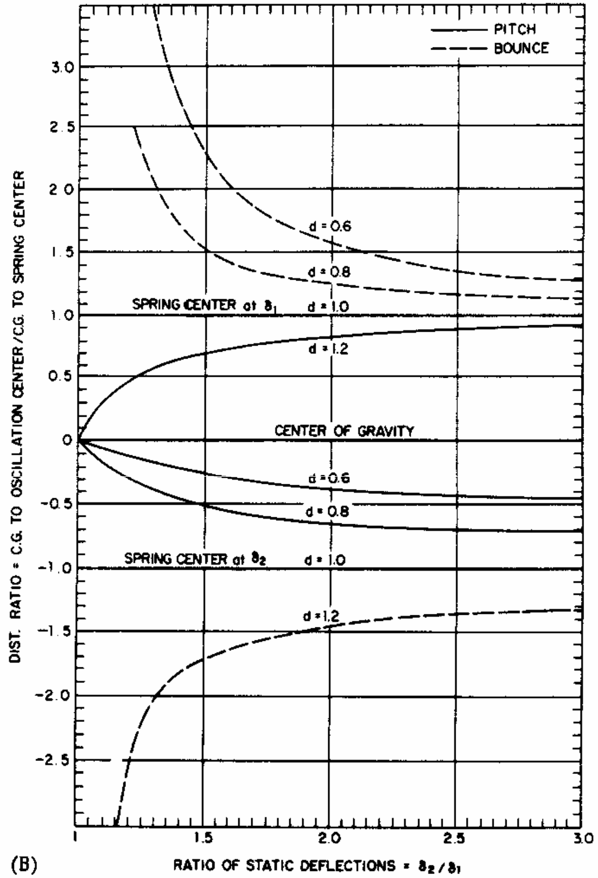
It will be seen in Fig. 14 that only the pitch frequency is sensitive to the suspension deflection ratio and to the dynamic index. However, both frequencies increase as the dynamic index diminishes and are a minimum when the static spring deflections are equal. When d = 1, each of the two modes consists of an independent vibration at one spring about the other spring as a center of oscillation.





(A)

Fig. 14 - Vibrating system with two degrees of freedom. Natural frequencies and centers of oscillation versus ratio of static deflections, δ_2/δ_1, and dynamic index, d



(B)

Fig. 14 - Continued

It is important to note that when d = 1, and the springs have the same static deflection ($\delta_2/\delta_1 = 1$), there are two other possible modes of vibration, since the two principal vibrations have the same frequency. One is a uniform bounce of the mass, when both springs act in the same phase, the other a pitch about the c.g. when both springs act in opposite phase. The curves of pitch center show that this condition exists whenever $\delta_2/\delta_1 = 1$, regardless of the dynamic index. This situation is definitely undesirable because it produces the maximum angular amplitude for a given vertical displacement at the springs. Consequently, this condition should be avoided in practice.

As long as there is a differential of static deflection between the two springs, a dynamic index of 1 gives the best vehicle ride because angular amplitudes are minimized and there is no interaction between springs.

2.7 References

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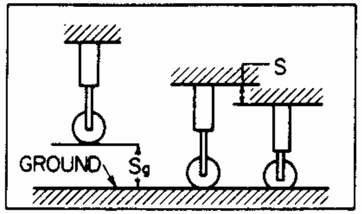
3. Energy Absorption and Impact

Figs. 15-17 are designed to facilitate the solution of problems involving the absorption of energy in bringing a moving mass to rest. Each chart provides an essential step in relating the resistance characteristic of the absorption gear to the displacement and resultant deceleration of the mass for any given initial velocity. These charts are particularly applicable to extreme operating conditions of vehicle suspensions and to normal operating conditions of airplane landing gears.



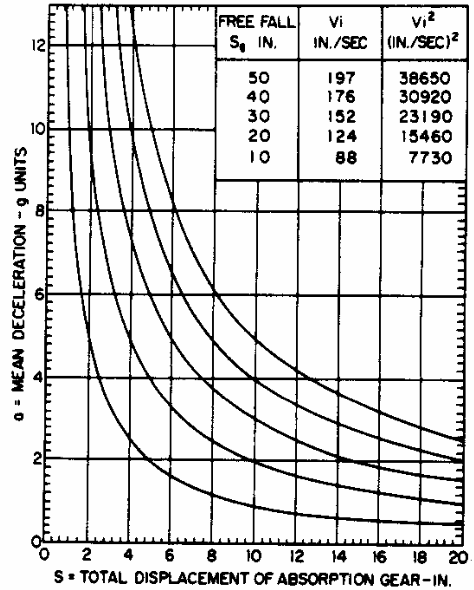


Fig. 15 - Mean deceleration versus displacement for constant impact velocity

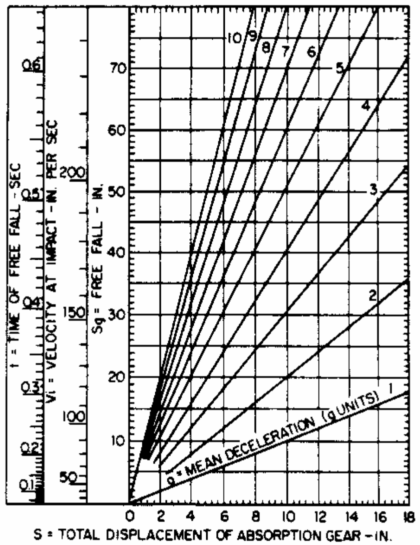


Fig. 16 - Freefall conditions

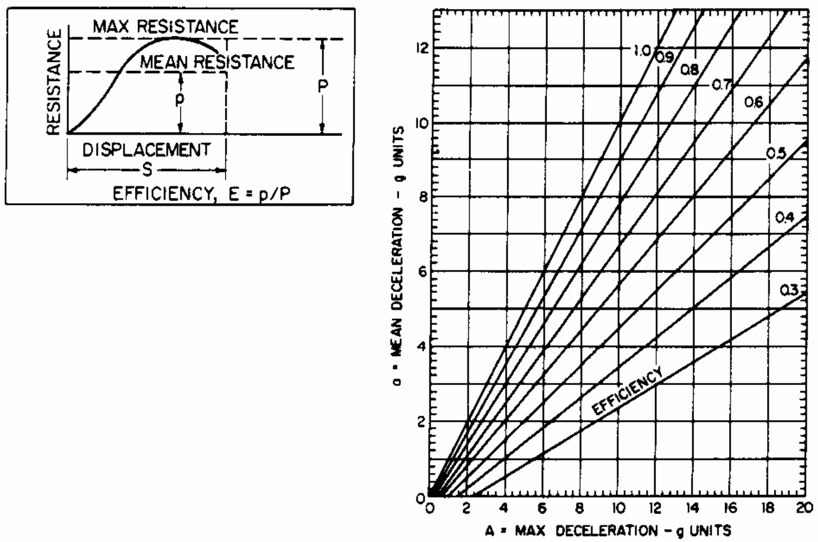


Fig. 17 - Freefall conditions, mean versus maximum deceleration at given efficiencies

Absorption systems usually combine the elastic resistance of springs with friction of damping forces, so that the resultant variation in resistance with displacement becomes complex. In order to make the charts readily applicable to any resistance characteristic, the resistance is reduced to a simple efficiency value. As shown diagrammatically in Fig. 17, the efficiency is taken as the ratio of mean to maximum resistance over the total displacement of the absorption gear.

The relations between velocity at impact, total displacement and mean deceleration, shown in Figs. 15 and 16, apply equally whether gravity acceleration is present or not. But, in relating the resistance of the absorption gear to the resultant deceleration, the force of gravity, if present, must be reckoned with as an additional component. Cases with and without gravity must, therefore, be treated differently as shown on the following pages. Where gravity is not involved the relations between mean and maximum deceleration and efficiency of the gear are simple and direct.

Note that, while Fig. 17 applies only to freefall conditions, it extends all the way to zero impact, i.e., a mass falling freely from rest in contact with the absorption gear. Since both the initial and final velocities are zero, the mean deceleration is also zero. Therefore, the maximum deceleration for any efficiency value is defined by the intersection of the constant efficiency line with the abscissa. Thus, Fig. 17 can be applied to a sprung mass, initially at rest but displaced from its static position.

3.1 List of Symbols

S_g = Distance of freefall to point of contact, in.
V_i = Velocity at impact, in./sec
t = Time of free fall, sec
S = Total displacement of absorption gear in bringing mass velocity to zero, in.
a = Mean deceleration of moving mass, g units
A = Maximum deceleration of moving mass, g units
p = Mean resistance of absorption gear over total displacement, lb
P = Maximum resistance of absorption gear, lb
E = Efficiency of absorption gear, p/P
W = Total moving weight, lb
g = 386 in./sec^2

3.2 Mathematical Relations

3.2.1 Impact from Free Fall

$V_i^2 = 2g\,S_g \qquad V_i = 27.8\sqrt{S_g}$

$t = \sqrt{2S_g/g} = 0.072\sqrt{S_g}$

$a = S_g/S$ or $a = V_i^2/(2gS)$

$a = (p/W) - 1 \qquad A = (P/W) - 1$

$E = p/P = (a+1)/(A+1)$

$a = E(A+1) - 1 \qquad A = [(a+1)/E] - 1$

Example:

Given: Freefall, $S_g = 40$ in.
Maximum displacement of gear, S = 10 in.
Maximum deceleration = 7g

To Find: Required efficiency of gear and maximum resistance

From Figs. 15 or 16 for $S_g = 40$ in. and S = 10 in. (read) a = 4g.

Required efficiency: From Fig. 17 for a = 4g and A = 7g (read) E = 62.5%.

Maximum resistance: A = (P/W) - 1 = 7
P = 8W

3.2.2 Impact Without Gravity Acceleration

$a = V_i^2/(2gS)$

$a = p/W \qquad A = P/W$

$E = p/P = a/A$

Example:

Given: Velocity at impact, $V_i = 176$ in./sec
Maximum Displacement of gear, S = 10 in.
Maximum Deceleration = 7g

To Find: Required efficiency of gear and maximum resistance

From Fig. 15, for $V_i = 176$ and S = 10 in. (read) a = 4g.

Required efficiency: E = p/P = a/A = 4/7 = 57.2%

Maximum resistance: P = AW = 7W

4. Vibration Limits for Passenger Comfort

A study of the subject indicates that there is no absolute standard of human comfort or discomfort expressed in physical terms such as amplitudes or acceleration at a given frequency. However, there is enough agreement among the test data from various investigators so that a zone may be outlined, above which vibration is certainly intolerable and below which it is immaterial. A limit of acceptable vibration can only be drawn by combining a judgment factor with analysis of the laboratory test data.

Three independent analyses appear in this section, each based on data from many investigators, and each with the author's own comments. The Burton-Douglas chart (Fig. 19) presents the conclusions of aircraft engineers, the Janeway recommendations (Fig. 20) are directed at automobile and railroad practice, and the Goldman presentation (Fig. 21) represents a broad biological viewpoint.

A comparison of these three presentations shows that above a frequency of 4 cps, the disagreement is minor, so that all analyses can be considered identical. Below this frequency, the presentations differ, partly because the data used are not entirely the same, but also because each author takes a different point of view based upon his particular field of interest. Note that all three are superimposed in Fig. 18 to show how they are related to each other in the 1-10 cps frequency range. It is significant that Janeway's recommended limits agree very closely with Goldman's mean discomfort threshold, while both fall well inside the very disagreeable threshold of the Burton-Douglas chart. For practical use, that chart should be selected which has been derived with the particular field of application in mind.

It will be noted that these criteria apply specifically to vertical vibration. For lateral vibrations, recently published experimental evidence from railway tests indicates that tolerable amplitudes for passenger comfort are 30% lower in the 1-2 cps frequency range.

The latest publications* on the subject strongly confirm the validity of constant peak jerk intensity as the comfort criterion for vertical vibration in the 1-6 cps frequency range. (See Fig. 20.)

*See Bibliography, Refs. 1, 4, 10, and 11.

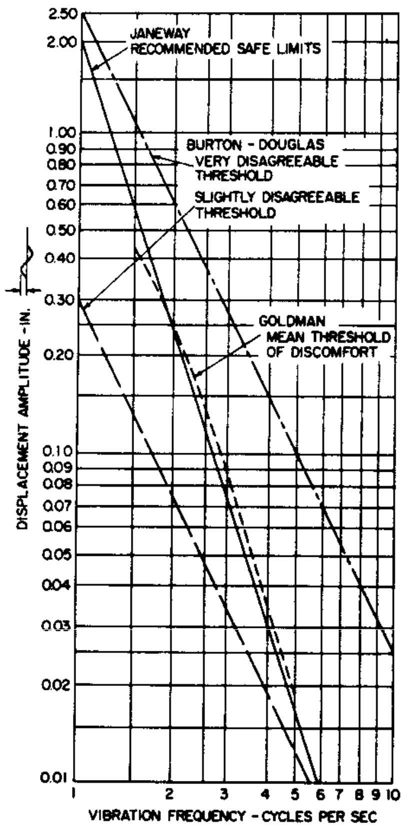


Fig. 18 - Discomfort thresholds — comparison at low frequencies

4.1 Response to Vertical Vibrations

E. F. Burton, Douglas Aircraft Co.

Passengers and crew in aircraft are subject to complex oscillatory disturbances, the effects of which must be evaluated in planning for comfort. The data on which such evaluations can be based are not extensive and are on sinusoidal oscillations, chiefly in a vertical plane, for a seated or standing person.

The accompanying chart (Fig. 19) is a simplified graph based on a careful comparison and evaluation of ten original studies ranging from motion

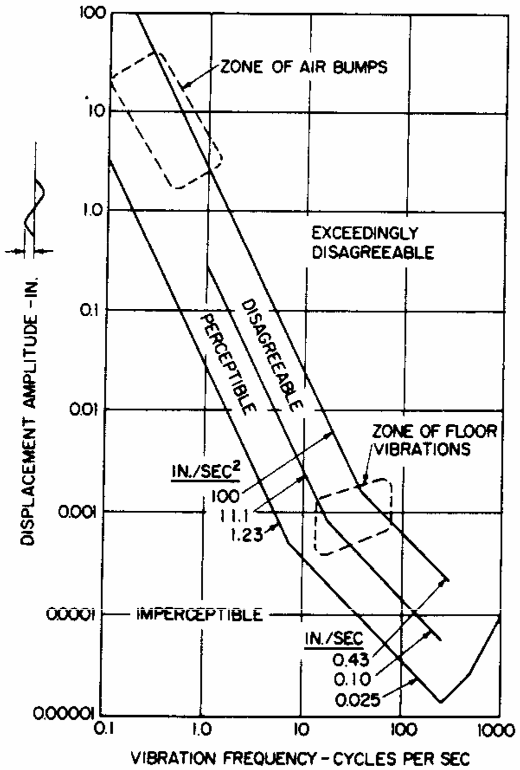


Fig. 19 - Response to vertical vibrations

sickness and ordinary vibration table studies down to fingertip sensitivity experiments. The assumption that the comfort classification is continuous over a wide range of frequencies and amplitudes, even though the physiological responses may differ within these ranges, permits the engineer to employ a single set of standards in judging the vibration experience in aircraft.

The graph applies to simple harmonic motions, but, if each major component of the complex steady-state vibration in an airplane is separately evaluated as if it were the only motion, a useful first approximation of the experience will be obtained. In the absence of other data, the graph may also serve as an approximate guide to the evaluation of fore and aft and lateral oscillations.

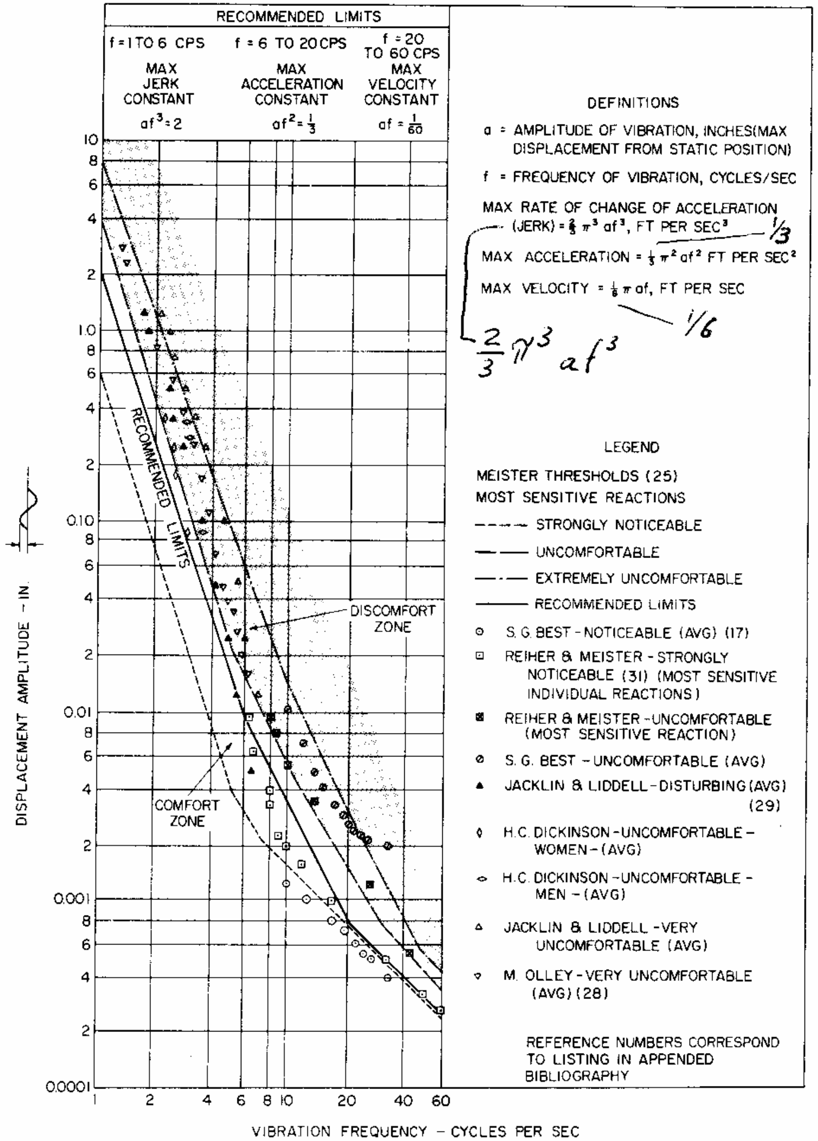


Fig. 20 - Human reaction to vertical vibration

The family of three curves shown in the graph is taken from Lippert*. Janeway's description in terms of maximum acceleration and maximum velocity is applied to each curve.

4.2 Vertical Vibration Limits for Passenger Comfort**

R. N. Janeway, Janeway Engineering Co.

In dealing with human reactions to vibration, limits for comfort cannot be fixed objectively, but must be derived by interpreting the available experimental data. The problem is complicated by the variations in individual sensitivity, and the diversity of test method and sensation level adopted by different investigators.

All the pertinent data available are correlated in Fig. 20 in terms of vibration amplitude versus frequency. The three broken lines define the sensation thresholds, respectively, of "strongly noticeable," "uncomfortable" and "very uncomfortable" for the most sensitive subjects at frequencies from 1 to 60 cycles per second, according to F. J. Meister***. Adjacent threshold lines encompass the results on at least 90% of all individuals tested and, therefore, also define the probable range of variation. The average results obtained by the other investigators confirm these boundaries remarkably well. The probable zones of "comfort" and "discomfort" are indicated on both sides of Meister's "uncomfortable" threshold.

The heavy line shows recommended limits which include a judgment factor and should be well within the comfort range even for the most sensitive person. It is important to note that the experimental data are all based on short exposure time of five minutes or less. Vibration, as actually encountered in vehicles, tends to be more prolonged the higher the frequency. For this reason, the recommended limits allow a greater margin of safety as the frequency increases.

As indicated at the top of Fig. 20, the recommended criterion consists of three simple relationships, each covering a portion of the frequency range, as shown in Table 2.

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*See Bibliography, Refs. 14 and 15.
**Based on Bibliography, Ref. 13.
***See Bibliography, Ref. 25.

TABLE 2

Frequency Range, cps	Amplitude versus Frequency, in
1-6	$af^3 = 2$ (constant peak jerk)
6-20	$af^2 = 1/3$ (constant peak acceleration)
20-60	$af = 1/60$ (constant peak velocity)

Example: At f = 2, the recommended amplitude limit is: a = 1/4 in.

The following should be noted in studying this figure:

1. All values are based on vertical sinusoidal vibration of a single frequency. Where two or more components of different frequencies are present, there is no established basis on which to evaluate the resultant effect. It is probable, however, that the component, which taken alone represents the highest sensation level, will govern the sensation as a whole.
2. A relatively low noise level is assumed so that the vibration alone determines the sensation. No data are available to indicate how to evaluate the resultant sensation where vibration and noise are of comparable intensities in terms of human reaction.
3. All data used were obtained with subjects standing (Refs. 25 and 31) or sitting on a hard seat. In any case where a cushion is used, it must be evaluated independently. Only the resultant vibration transmitted to the passenger should be related to the chart values.

4.3 Subjective Responses of the Human Body to Vibratory Motion*

David E. Goldman, Naval Medical Research Inst.

Fig. 21 shows mean amplitudes of simple harmonic vibration at frequencies up to 60 cycles per sec, at which subjects: I) perceive vibration, II) find it unpleasant, or III) refuse to tolerate it.

Each point on the chart represents an average of 4-9 different values based on measurements reported by various investigators. The data are for short duration exposure (a few minutes) of the body standing or sitting on a vibrating support subjected to vertical oscillation.

*Based on Bibliography, Ref. 12.

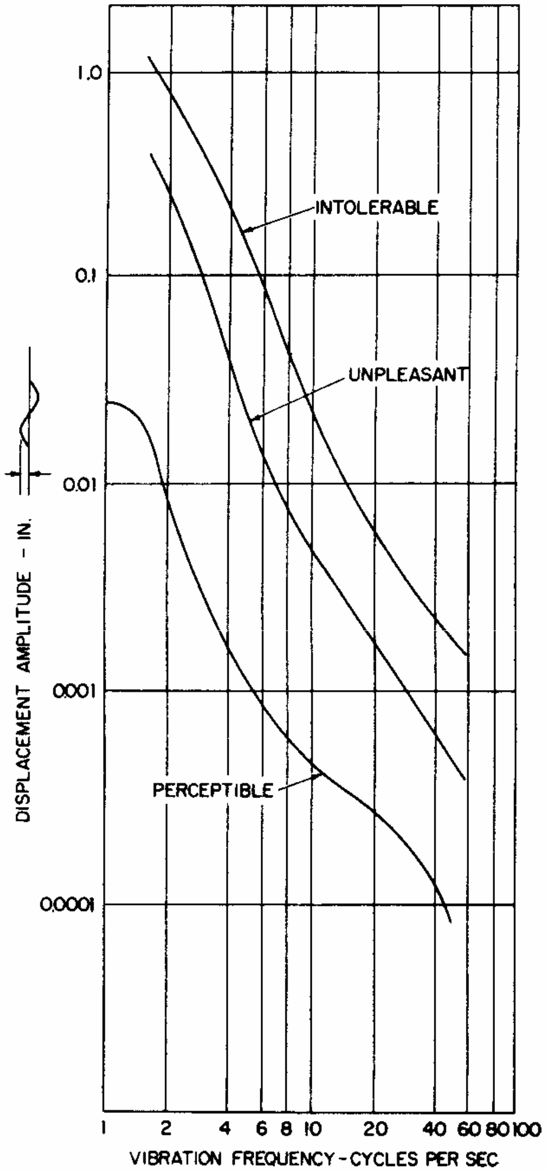

Fig. 21 - Subjective response of the human body to vibratory motion

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